Quasi-Random Hypergraphs and Extremal Problems for Hypergraphs

D I S S E R T A T I O N

zur Erlangung des akademischen Grades

doctor rerum naturalium
(Dr. rer. nat.)
im Fach Informatik

eingereicht an der
Mathematisch-Naturwissenschaftlichen Fakultät II
Humboldt-Universität zu Berlin

von
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eingereicht am: 23.06.2010
Tag der mündlichen Prüfung: 22.11.2010

to my mother

Zusammenfassung

Das Regularitätslemma ist ein zentrales Werkzeug aus der Extremalen Graphentheorie mit Anwendungen in der Additiven Zahlentheorie, der Diskreten Geometrie und der Theoretischen Informatik. Dieses Lemma war ein zentraler Hilfssatz in Szemerédis Beweis der zahlentheoretischen Vermutung von Erdős und Turán, dass jede Teilmenge der natürlichen Zahlen mit positiver oberer Dichte arithmetische Progressionen beliebiger endlicher Länge enthält.

Das Regularitätslemma besagt, dass man die Knotenmenge jedes Graphen in konstant viele fast gleich große Teilmengen partitionieren kann, so dass die meisten auf je zwei solcher Teilmengen induzierten bipartiten Graphen quasi-zufällig sind. Die systematische Theorie quasi-zufälliger Graphen wurde von Thomason initiiert und etwas später haben Chung, Graham und Wilson verschiedene Eigenschaften quasi-zufälliger Graphen studiert und deren Äquivalenz im approximativen Sinne bewiesen. Im Weiteren wurde die Untersuchung der Quasi-Zufälligkeit auf andere diskrete Strukturen ausgedehnt. Heutzutage existieren mehrere Regularitätslemmata für Graphen und Hypergraphen, die deren Zerlegung in quasi-zufällige Teile garantieren. Dabei werden unterschiedliche Ausprägungen von quasi-zufälligen Eigenschaften zu Grunde gelegt.

In dieser Arbeit wird zuerst das Theorem von Chung, Graham und Wilson über quasi-zufällige Graphen zur sogenannten schwachen Quasi-Zufälligkeit für k-uniforme Hypergraphen verallgemeinert und somit eine Reihe äquivalenter Eigenschaften bestimmt. Basierend auf diesen Resultaten werden nichtbipartite Graphen gefunden, welche die Quasi-Zufälligkeit für Graphen "forcieren". Zuvor waren nur bipartite Graphen mit dieser Eigenschaft bekannt. Desweiteren ist ein konzeptionell einfacher Algorithmus zum Verifizieren nicht erfüllbarer zufälliger k-SAT Formeln angegeben.

Dann richtet sich der Fokus auf Anwendungen verschiedener Regularitätslemmata für Hypergraphen. Zuerst wird die Menge aller bezeichneten 3-uniformen Hypergraphen auf n Knoten, die keine Kopie des Hypergraphen der Fano Ebene enthalten, studiert. Es wird gezeigt, dass fast jedes Element aus dieser Menge ein bipartiter Hypergraph ist. Dies führt zu einem Algorithmus, der in polynomiell erwarteter Zeit einen zufälligen Fano-freien (und somit einen zufälligen bipartiten 3-uniformen) Hypergraphen richtig färbt.

Schließlich wird die folgende extremale Funktion studiert. Es sind r Farben gegeben sowie ein k-uniformer Hypergraph F. Auf wie viele verschiedene Arten kann man die Kanten eines k-uniformen Hypergraphen H färben, so dass keine monochromatische Kopie von F entsteht? Welche Hypergraphen H maximieren die Anzahl erlaubter Kantenfärbungen? Hier wird ein strukturelles Resultat für eine natürliche Klasse von Hypergraphen bewiesen. Es wird für viele Hypergraphen F, deren extremaler Hypergraph bekannt ist, gezeigt, dass im Falle von zwei oder drei Farben die extremalen Hypergraphen die oben beschriebene Funktion maximieren, während für vier oder mehr Farben andere Hypergraphen mehr Kantenfärbungen zulassen.

Abstract

The regularity lemma was originally developed by Szemerédi in the seventies as a tool to resolve a long standing conjecture of Erdős and Turán, that any subset of the integers of positive upper density contains arbitrary long arithmetic progressions. Soon this lemma was recognized as an important tool in extremal graph theory and it also has had applications to additive number theory, discrete geometry and theoretical computer science. It roughly says that one can partition a vertex set of any graph into constantly many parts almost all of which look random-like. This random-like behaviour is referred to as quasi-randomness. More generally, the systematic study of quasi-random graphs was initiated by Thomason and, subsequently, Chung, Graham and Wilson collected several disparate properties of random graphs that all turned out to be equivalent in a deterministic sense. Later on, quasi-random properties were studied for various discrete structures.

This thesis presents first one possible generalization of the result of Chung, Graham and Wilson to k-uniform hypergraphs, and studies the so-called weak quasi-randomness. As applications we obtain a simple strong refutation algorithm for random sparse k-SAT formulas and we identify first non-bipartite forcing pairs for quasi-random graphs.

Our focus then shifts from the study of quasi-random objects to applications of different versions of the hypergraph regularity lemmas; all these versions assert decompositions of hypergraphs into constantly many quasi-random parts, where the meaning of “quasi-random” takes different contexts in different situations.

We study the family of hypergraphs not containing the hypergraph of the Fano plane as a subhypergraph, and show that almost all members of this family are bipartite. As a consequence an algorithm for coloring bipartite 3-uniform hypergraphs with average polynomial running time is given.

Then the following combinatorial extremal problem is considered. Suppose one is given r colors and a fixed hypergraph F. The question is: In at most how many ways can one color the hyperedges of a hypergraph H on n vertices such that no monochromatic copy of F is created? What are the extremal hypergraphs for this function? Here a structural result for a natural family of hypergraphs F is proven. For some special classes of hypergraphs we show that their extremal hypergraphs (for large n) maximize the number of edge colorings for 2 and 3 colors, while for at least 4 colors other hypergraphs are optimal.

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1 Introduction

1.1 Background

The celebrated theorem of Szemerédi [Sze75] states that any subset of the integers of positive upper density contains arbitrary long arithmetic progressions. One of the key ingredients in his proof was a lemma that was later coined the regularity (or uniformity) lemma [Sze78]. The regularity lemma asserts that one can partition the vertex set of any graph into constantly many subsets of nearly equal size so that the bipartite graph that is induced by any two subsets almost always looks like a random bipartite graph. Though the regularity lemma was originally intended to prove theorems in number theory, it soon became one of the central tools in extremal graph theory and beyond, as its applications and further generalizations led to results in combinatorial geometry, additive number theory, random graph theory and more recently in theoretical computer science in the area of property testing.

Nowadays one might also say that the regularity lemma decomposes the edge set of any graph into constantly many bipartite graphs, almost all of which are quasi-random, where quasi-randomness stands for some "deterministic concept" of randomness. Szemerédi used the term "ε-regular pair" to denote a bipartite quasi-random graph. The systematic study of quasi-random graphs was initiated by Thomason [Tho87a, Tho87b], and, subsequently, Chung, Graham and Wilson [CGW89] built on the work of others to show that several seemingly unrelated properties of $G(n,p)$ are equivalent in a deterministic sense.

In the following years further quasi-random discrete structures were investigated such as quasi-random hypergraphs, tournaments, subsets of $\mathbb{Z}_N$ and more generally set systems, functions and groups [Chu90, CG91, SS91, CG92b, Gow07, Gow08].

Gowers [Gow01] exploited different characterizations of quasi-random subsets of $\mathbb{Z}_N$ to give a new proof of Szemerédi's theorem with better upper bounds on the density of a subset $A \subset \{1, \ldots, n\}$ that does not contain an arithemitc progression of length k.

Another recent proof of Szemerédi's theorem uses a strong generalization of the regularity lemma to uniform hypergraphs, which was a result of a programme carried out by Rödl and his collaborators [FR02, RS04, NRS06a, RS07b, RS07a]. Independently, Gowers proved a hypergraph regularity lemma of similar strength [Gow06, Gow07]. Both approaches assert that the hyperedge set of any hypergraph can be decomposed into constantly many quasi-random pieces.

Earlier in the 90's, Steger [Ste90] and Chung [Chu91] independently observed that a straightforward generalization of the regularity lemma for graphs holds for hypergraphs. However, the quasi-random properties provided by an application of this lemma are weaker compared to those of Gowers and Rödl et al. (and therefore this lemma will be referred to as the weak hypergraph regularity lemma). Thus, for example, this lemma does not imply Szemeredi's theorem. Still, quite recently, more applications of the lemma also appeared. Different concepts of quasi-randomness are used in different versions of the hypergraph regularity lemmas.

The original proof of the regularity lemma was non-constructive. Alon, Duke, Lefmann, Rödl and Yuster found a way to make the regularity lemma algorithmic [ADL$^+$94]. Their algorithm has a running time $O(M(n))$, where $M(n) = O(n^{2.376})$ is the time needed to multiply two $n \times n$ matrices with entries from $\{0, 1\}$ over the integers. In some sense it is a rather surprising result since deciding whether a given partition is ε-regular is co-NP-complete. On the other hand, different characterizations (which are equivalent in an approximate way) of quasi-random graphs are a way to get around this issue. For hypergraphs, Czygrinow and Rödl [CR00] gave a first algorithmic version of the so-called weak hypergraph regularity lemma. An algorithmic version of a strong hypergraph regularity lemma for 3-uniform hypergraphs with a corresponding counting lemma was developed by Haxell, Nagle and Rödl [HNR08]; see also [DHNR02] and a discussion on the equivalences of various notions of relative hypergraph quasi-randomness by Nagle, Poerschke, Rödl and Schacht [NPRS09]. Applications of algorithmic regularity lemmas often lead to efficient (from a theoretical point of view) approximation algorithms. Most notably however, the role of the regularity lemmas lies at the heart of the area of property testing, thus connecting extremal graph theory with theoretical computer science.

This thesis studies some natural hypergraph quasi-randomness concepts and applications of hypergraph regularity lemmas to problems in extremal combinatorics. As consequences, we find a simple strong refutation algorithm for random sparse k-SAT formulas, we identify first non-bipartite forcing pairs, and we develop an algorithm with polynomial average running time that colors every bipartite 3-uniform hypergraph properly.

1.2 Main results: a guide to the thesis

Below we summarize the main results presented in this thesis. In the next chapter, we review some basic notation and tools. We introduce the regularity lemma for graphs and then its generalizations to hypergraphs. Here we will discuss the weak and the strong regularity lemmas for hypergraphs. In Chapter 3 we develop the theory of weak quasi-random hypergraphs. Then in Chapters 4 and 5 we apply the hypergraph regularity lemmas (mostly compatible with the concept of quasi-randomness studied in Chapter 3) to solve problems in extremal combinatorics. Chapters 3, 4 and 5 contain concluding remarks and possible future directions for further research.

1.2.1 Weak quasi-randomness for uniform hypergraphs

The systematic study of quasi-random or pseudo-random graphs was initiated by Thomason [Tho87a, Tho87b], who studied deterministic graphs G_n of density p that "imitate" the binomial random graph $G(n, p)$, i.e., graphs G_n that share some important properties with $G(n,p)$. One of the key properties of $G(n,p)$ is its uniform edge distribution and Thomason chose a quantitative version of this property, so-called *jumbledness*, to define pseudo-random graphs. Subsequently, Chung, Graham and Wilson [CGW89] (building on the work of others) considered a variation of jumbledness and showed that several other seemingly unrelated properties of $G(n,p)$ are equivalent to it in a deterministic sense.

In Chapter 3, we study quasi-random properties of k-uniform hypergraphs. As mentioned above, the following beautiful theorem may be seen as a starting point for the theory of quasi-random graphs:

Theorem 1.1 (Chung, Graham and Wilson [CGW89])**.** *For any sequence $(G_n)_{n\in\mathbb{N}}$ of graphs with $|V(G_n)| = n$, the following properties are equivalent:*

P_1 : *for all graphs F, we have $N^*_F(G_n) = (1/2)^{\binom{\ell}{2}} n^\ell + o(n^\ell)$, where $\ell = |V(F)|$ and $N^*_F(G_n)$ denotes the number of labeled, induced copies of F in G_n;*

P_2 : *$e(G_n) \geq \frac{1}{2}\binom{n}{2} - o(n^2)$ and $N_{C_4}(G_n) \leq (n/2)^4 + o(n^4)$, where C_4 is the cycle on 4 vertices and $N_{C_4}(G)$ denotes the number of labeled (not necessarily induced) copies of C_4 in G_n;*

P_3 : *$e(G_n) \geq \frac{1}{2}\binom{n}{2} - o(n^2)$, $\lambda_1(G_n) = n/2 + o(n)$, and $|\lambda_2(G_n)| = o(n)$, where $\lambda_i(G_n)$ is the i-th largest eigenvalue of the adjacency matrix of G_n in absolute value;*

P_4 : *for every subset $U \subseteq V(G_n)$, we have $e(U) = \frac{1}{2}\binom{|U|}{2} + o(n^2)$;*

P_5 : *for every subset $U = \lfloor n/2 \rfloor$, we have $e(U) = n^2/16 + o(n^2)$;*

P_6 : *$\sum_{u,v} |s(u,v) - n/2| = o(n^3)$, where for vertices $u, v \in V(G_n)$ we set*

$$s(u,v) = |\{x \in V(G_n)\colon\ ux \in E(G_n) \Leftrightarrow vx \in E(G_n)\}|;$$

P_7 : *$\sum_{u,v} |\mathrm{codeg}(u,v) - n/4| = o(n^3)$, where for vertices $u, v \in V(G_n)$ we set*

$$\mathrm{codeg}(u,v) = |\{x \in V(G_n)\colon\ ux \in E(G_n) \text{ and } vx \in E(G_n)\}|. \quad \square$$

Note that the property P_4 implies the density of G_n must tend to $1/2$ as n tends to infinity. However, the properties $P_1, \ldots, P_7$ can be altered in a straightforward way and the analogue of Theorem 1.1 holds for all fixed, positive densities. Thus, graphs satisfying one (and hence all) of the properties $P_1, \ldots, P_7$ are called *quasi-random* and $P_1, \ldots, P_7$ are referred to as quasi-random properties.

Another result related to our work in Chapter 3 is the following, which is due to Simonovits and Sós [SS97].

Theorem 1.2 (Simonovits and Sós)**.** *For every $d > 0$, every graph F on ℓ vertices containing at least one edge, and every $\varepsilon > 0$ there exist $\delta > 0$ and n_0 such that the following is true. If $G = (V, E)$ is a graph with $|V| = n \geq n_0$ vertices such that $N_F(U) = d^{e(F)}|U|^\ell \pm \delta n^\ell$ for every subset $U \subseteq V$, where $N_F(U)$ denotes the number of* *labeled copies of F in the induced subgraph $G[U]$, then $e(U) = d\binom{|U|}{2} \pm \varepsilon n^2$ for every subset $U \subseteq V$.* □

This result states, that if every large induced subgraph of G contains approximately the "right" density of some fixed nonempty graph F, then G is quasi-random. On the other hand, this property, which we will refer to as hereditary, is implied by any of the properties $P_1, \ldots, P_7$. There are further strengthenings of the properties discussed above, see, e.g., [Sha10, SY08, SS91, SS97, SS03, ST04, Sha10, SYa, Yus08].

Haviland and Thomason [HT89, HT92] generalizing concepts of Thomason [Tho87a, Tho87b], introduced jumbledness for hypergraphs, where one says that a k-uniform hypergraph H is (p, α)-jumbled, if for every $U \subseteq V(H)$ one has

$$\left| e_H(U) - p\binom{|U|}{k} \right| \leq \alpha |U|.$$

There, the authors studied conditions on degrees and co-degrees of the vertices enforcing jumbledness. This concept was too weak to imply a property for hypergraphs similar to P_1. For example there are hypergraphs H, which are $(p, o(n^{k-1}))$-jumbled, but do not contain the "right" number of copies of certain hypergraphs. Some of these hypergraphs do not contain even any copy of some small hypergraphs. Examples were given by several researchers in [HT89, CG90, FR88]. One of them can be traced back to the work of Erdős and Hajnal [EH72]. It is the 3-uniform hypergraph consisting of directed triangles of a random tournament. This hypergraph is easily seen to be $(1/4, o(n^2))$-jumbled, but does not contain even 3 hyperedges on any set of 4 vertices. This shows that there is no

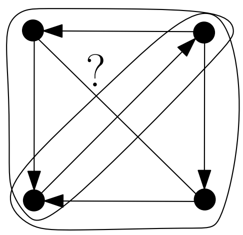

Figure 1.1: The hypergraph of a tournament

straightforward extension of quasi-randomness to k-uniform hypergraphs.

Chung and Graham [Chu90, CG90] and Kohayakawa, Rödl, and Skokan [KRS02] studied generalizations of some of the properties $P_1, \ldots, P_7$ to hypergraphs and showed their equivalences. Roughly speaking, a k-uniform hypergraph H_n of density d is quasi-random, in their sense, if the edges in H_n intersect a d-proportion of the cliques of order k of every $(k-1)$-uniform hypergraph on the same vertex set. In fact, this property can be viewed as a generalization of P_4 and as it turned out, this notion of quasi-randomness implies the natural analogue of P_1 for k-uniform hypergraphs. On the other hand, there exists no appropriate extension of Szemerédi's regularity lemma [Sze78], i.e., there exists no lemma, which guarantees a decomposition of any given k-uniform hypergraph into relatively "few" blocks, such that most of them satisfy this notion of quasi-randomness.

However, a variation of this notion, a so-called relative quasi-randomness, together with a corresponding regularity lemma for k-uniform hypergraphs was found by Gowers [Gow06, Gow07] and Rödl et al. [FR02, RS04, NRS06a, RS07b, RS07a]; see also [NPRS09] on the equivalence of this notion for 3-uniform hypergraphs.

We consider extensions of Theorem 1.1 and Theorem 1.2 to k-uniform hypergraphs. We study a simpler concept of uniform edge distribution, which only enforces similar densities induced on vertex sets, which is $(p, o(n^{k-1}))$-jumbledeness for k-uniform hypergraphs. More precisely, we consider the following straightforward extension of P_4.

$\mathrm{DISC}_d(\delta)$ We say a k-uniform hypergraph H_n on n vertices has $\mathrm{DISC}_d(\delta)$ for d, $\delta > 0$, if

$$e(U) = d\binom{|U|}{k} \pm \delta n^k \quad \text{for all} \quad U \subseteq V(H_n)\,,$$

where by $x = y \pm z$ we mean that x lies in the interval $[y - z, y + z]$.

Hypergraphs H with property DISC_d were studied in [Chu90, KNRS10] and a straightforward generalization of Szemerédi's regularity lemma for this concept was observed to hold in [Chu91, FR92, Ste90] (see Theorem 2.7 in Chapter 2). Moreover, it was shown in [KNRS10] that the property $\mathrm{DISC}_d(\varepsilon)$ for H with $\varepsilon \ll d$ implies that H contains approximately the right number of copies of any fixed *linear* hypergraph, i.e. any two of its hyperedges intersect in at most one vertex. Indeed, if we define $H := \mathcal{K}_k(\mathcal{G}(n, p))$, that is, it consists of hyperedges being the cliques of size k in the random graph $\mathcal{G}(n, p)$, then a few lines of calculations show that H will satisfy $\mathrm{DISC}_d(o(1))$, where $d = p^{\binom{k}{2}}$, with exponentially high probability, but it will contain the "right" number of copies only of linear hypergraphs.

We will suggest extensions of properties P_1, P_2, P_6, and P_7 to k-uniform hypergraphs which all turn out to be equivalent to DISC_d (the analogue of P_4 in this context). More background and ideas behind the generalizations are provided in the first section of Chapter 3, as the somewhat technical nature of the generalization requires a bit of notation. The properties which we will identify will be called CL_d, ICL_d, MIN_d, DEV_d, and MDEG_d, see Theorem 1.3 and its discussion in Section 3.1.1.

Here we discuss only one property, called MIN_d, which is a generalization of P_2. It is not difficult to see that a graph of edge density d has at least $d^4n^4 + o(n^4)$ copies of C_4, the cycle on 4 vertices. So property P_2 says that graphs attaining this minimum are quasi-random. For the notion of quasi-randomness studied by Chung and Graham [CG90], the role of C_4 was taken over by the k-uniform octahedron, which is the complete k-partite k-uniform hypergraph with class sizes 2. In our case, having the "right" number of octahedrons implies DISC_d, but it is not equivalent to it, as octahedrons are not linear for $k \geq 3$. So we have to find a linear hypergraph, whose number of occurrences together with the assumption on the density would imply DISC_d. This hypergraph will be denoted by M_k and we give here for brevity only its geometric characterization: its vertex set corresponds to the edges of the k-dimensional hypercube and its hyperedges consist of those edges that all meet in one vertex. Thus, M_k has $k2^{k-1}$ vertices and 2^k hyperedges. We formulate the generalization of P_2 as follows:

$\mathrm{MIN}_d(\varepsilon)$ We say a k-uniform hypergraph H_n on n vertices has $\mathrm{MIN}_d(\varepsilon)$ for d, $\varepsilon > 0$, if

$$e(H_n) \geq d\tbinom{n}{k} - \varepsilon n^k \quad \text{and} \quad N_{M_k}(H_n) \leq d^{2^k} n^{k2^{k-1}} + \varepsilon n^{k2^{k-1}} .$$

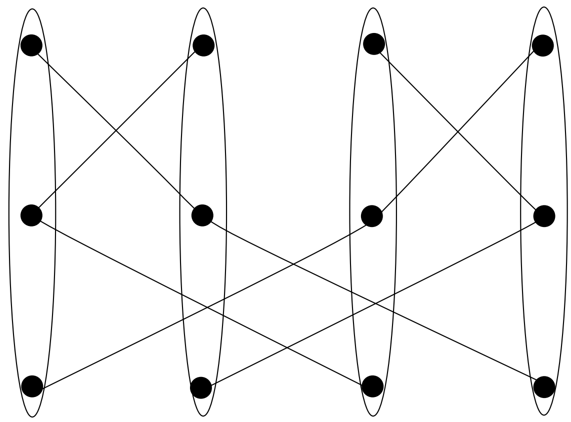

Figure 1.2: The hypergraph M_3: ovals and curves represent its hyperedges.

The following theorem may be seen as one possible generalization of Theorem 1.1 to uniform hypergraphs.

Theorem 1.3. *For every integer $k \geq 2$ and every $d > 0$ the properties* DISC_d*,* CL_d*,* ICL_d*,* MIN_d*,* DEV_d*, and* MDEG_d *are equivalent.*

We will also verify the equivalence of another property for k-uniform hypergraphs, which is inspired by Theorem 1.2 and which we discuss in Section 3.1.3 of Chapter 3 (see Theorem 3.3). Then we show the equivalence of several partite variants of DISC_d (see Theorem 3.4 in Section 3.1.4).

Forcing pairs

Another interesting topic related to quasi-random graphs is forcing pairs of graphs. Property P_2 essentially says that if the density of a graph G is at least $d - o(1)$ and the density of 4-cycles is at most $d^4 + o(1)$, then G is a quasi-random graph with density d. In other words, lower and upper bounds on the number of K_2 and C_4 in G imply that G is quasi-random and the question arises which other pairs of graphs replacing K_2 and C_4 have the same effect. Such pairs are called *forcing pairs* (cf. notion of forcing pairs in [CGW89]; in [CG91] note that it refers to induced densities). For example, it was noticed in [CGW89] and [ST04] that C_4 may be replaced by any even cycle or any complete bipartite graph $K_{a,b}$ with $a, b \geq 2$. Moreover, it follows from the recent work of Hatami [Hat10] that C_4 can be replaced by Q_k, the graph of the k-dimensional hypercube for $k \geq 2$.

However, all those forcing pairs consist of bipartite graphs and it would be interesting to find forcing pairs involving non-bipartite graphs (see, e.g., [SY08]). Using

Theorem 1.2 together with Theorem 1.3 we identify first forcing pairs of non-bipartite graphs, described below. For an integer k let $M(k)$ be the graph which we obtain if we replace every hyperedge of the k-uniform hypergraph M_k by a graph clique of order k. Since the k-uniform hypergraph M_k is linear, the graph $M(k)$ consists of 2^k graph cliques K_k, which intersect in at most one vertex. Hence, $M(k)$ consists of $k2^{k-1}$ vertices and $2^k\binom{k}{2}$ edges. (Alternatively, $M(k)$ is the graph we obtain from the k-dimensional hypercube, by letting $V(M(k))$ be the edges of the hypercube and letting edges of $M(k)$ connect two edges of the hypercube if they have a common end-vertex. In other words, $M(k)$ is the line graph of the graph of the k-dimensional hypercube Q_k.) The following corollary of Theorem 1.3 shows that for every $k \geq 2$ the pair of graphs K_k and $M(k)$ is a forcing pair.

Corollary 1.4. *For every integer $k \geq 2$, every $d > 0$, and every $\delta > 0$ there exist $\varepsilon > 0$ and n_0 such that the following is true. If $G = (V, E)$ is a graph on $|V| = n \geq n_0$ vertices that satisfies*

$$N_{K_k}(G) \geq d^{\binom{k}{2}} n^k - \varepsilon n^k \quad \textit{and} \quad N_{M(k)}(G) \leq d^{2^k\binom{k}{2}} n^{k2^{k-1}} + \varepsilon n^{k2^{k-1}} ,$$

then G satisfies $\mathrm{DISC}_d(\delta)$.

A simple strong refutation algorithm

Let $X_n = \{x_1, \dots, x_n\}$ be a set of n propositional variables and $L = \{x_i, \bar{x_i} \mid i \in [n]\}$ be the set of literals. A k-clause is a disjunction of k literals, and a k-SAT formula is a conjunction of some k-clauses. The k-SAT problem is to decide whether a given k-SAT formula F is satisfiable, i.e. if there exists an assignment $\varphi\colon X_n \to \{0, 1\}$ such that every clause is satisfied. It is among the best studied NP-complete problems. Moreover, the MAX-k-SAT problem, asking for the maximum possible number of satisfiable clauses cannot be approximated in polynomial time by a factor $1 - 1/2^k + \varepsilon$ for any fixed $\varepsilon > 0$ unless P=NP, as shown by Håstad [Hås01]. On the other hand, a simple probabilistic argument shows that the number of unsatisfiable clauses for a k-SAT formula F is at most $2^{-k}|F|$, where $|F|$ denotes the number of clauses.

Here we are motivated by the research on strong refutation heuristics studied first by Coja-Oghlan, Goerdt, and Lanka [COGL07]. We consider the following problem. Let $p = p(n) \in [0, 1]$, and let $\mathcal{F}_k(n, p)$ be the probability space over all k-SAT formulas on X_n, for which each of the $(2n)^k$ possible (ordered) k-clauses will be included independently with probability p. An algorithm is a strong refutation algorithm if w.h.p. for $F \in \mathcal{F}_k(n, p)$ it approximates $\mathrm{unsat}(F)$ by a factor of $(1 - \varepsilon)$ and never outputs a number bigger than $\mathrm{unsat}(F)$, where $\mathrm{unsat}(F)$ is the minimum number of unsatisfied clauses in F over all possible assignments. From a strong refutation algorithm we demand that it verifies this bound on $\mathrm{unsat}(F)$. One is interested in polynomial time strong refutation algorithms.

Definition 1.5. *Let $k \geq 3$, $\varepsilon > 0$, and $p = p(n)$. An algorithm $\mathcal{A}$ is an ε-*strong refutation algorithm *for $\mathcal{F}_k(n, p)$ if for a given k-SAT formula F on X_n the algorithm $\mathcal{A}$ outputs an integer $\mathcal{A}(F)$ such that* (*i*) $\mathcal{A}(F) \leq \text{unsat}(F)$ *and*

(*ii*) $\lim_{n\to\infty} \mathbb{P}(\mathcal{A}(F) \geq (1-\varepsilon)\text{unsat}(F)) = 1$ *for* $F \in \mathcal{F}_k(n,p)$.

It follows from Chernoff's inequality (Theorem 2.19) that for $p \gg n^{1-k}$ w.h.p.

$$\text{unsat}(F) = (2^{-k} + O(\sqrt{n^{-k+1}p^{-1}}))|F| = (2^{-k} + o(1))|F| \tag{1.1}$$

for $F \in \mathcal{F}_k(n,p)$. Therefore note that for $p \gg n^{1-k}$ the trivial algorithm, which returns $(2^{-k} - \varepsilon)|F|$ for every F, satisfies condition (*ii*), but fails to fulfill (*i*).

Refutation and strong refutation algorithms were studied by several researchers and the best strong refutation algorithms for $k = 3, 4$ are due to Coja-Oghlan, Goerdt, and Lanka [COGL07] and for general $k \geq 5$ are due to Coja-Oghlan, Cooper, and Frieze [COCF09] (see also [Fei02, FO07, FKO06]). Those authors found ε-strong refutation algorithms for every $\varepsilon > 0$ and $p \gg p_k$, where

$$p_k = \begin{cases} n^{-1.5}(\log n)^6 & \text{if } k = 3, \\ n^{-2} & \text{if } k = 4, \\ n^{-\lfloor k/2 \rfloor} & \text{if } k \geq 5. \end{cases}$$

The algorithms from [COGL07] and [COCF09] rely on tools from linear algebra. More precisely, the authors [COGL07] define some auxiliary graphs and perform a clever spectral analysis of their adjacency matrices, which leads to some conclusion about the discrepancy in hypergraphs. These hypergraphs correspond to formulas from $\mathcal{F}_k(n,p)$. We however, will take a direct approach and instead of dealing with auxiliary graphs, we will make use of the DEV property from Theorem 1.3 together with some concentration results about homomorphism occurences of M_k in a random k-uniform k-partite hypergraph. An advantage of our approach are elementary ε-strong refutation algorithms for every $k \geq 3$ for $p \gg n^{-(k-1)/2}$.

Theorem 1.6. *For every $k \geq 3$, $\varepsilon > 0$, and $o(1) = p(n) \gg n^{-(k-1)/2}$ there is an ε-strong refutation algorithm for $\mathcal{F}_k(n,p)$ with running time $O(n^{k2^{k-1}})$ independent of ε.*

The application of weak quasi-randomness to refutation algorithms is joint work with Hiệp Hàn and Mathias Schacht [HPS09], while all other results of the Chapter 3 were obtained with David Conlon, Hiệp Hàn and Mathias Schacht [CHPS].

1.2.2 Almost all hypergraphs without Fano planes are bipartite and its algorithmic consequences

For a hypergraph L, we denote by $\text{Forb}(n, L)$ the family of all L-free labeled hypergraphs with the vertex set $[n]$. As a lower bound on its cardinality, we have

$$|\text{Forb}(n, L)| \geq 2^{\text{ex}(n,L)},$$

where $\mathrm{ex}(n, L)$, called *extremal* number, is the maximum number of edges a hypergraph on n vertices can have without containing a copy of L as a subgraph. The above lower bound is trivial, as any subgraph of an extremal hypergraph for L is L-free.

For graphs, following the work of Kleitman and Rothschild [KR75] on posets, the question of estimating $|\mathrm{Forb}(n, L)|$, for a fixed graph L, was first studied by Erdős, Kleitman, and Rothschild [EKR76]. In particular, those authors proved that almost every triangle-free graph is bipartite. Moreover, they showed for $p \geq 2$

$$|\mathrm{Forb}(n, K_{p+1})| \leq 2^{(1-\frac{1}{p})\binom{n}{2}+o(n^2)}. \tag{1.2}$$

Thus, the dominating term in the exponent turns out to be the extremal number $\mathrm{ex}(n, K_{p+1})$. Later, Erdős, Frankl, and Rödl [EFR86] extended this result from cliques to arbitrary graphs L with chromatic number $\chi(L) \geq 3$, by proving

$$|\mathrm{Forb}(n, L)| \leq 2^{(1+o(1))\mathrm{ex}(n,L)}. \tag{1.3}$$

A strengthening of (1.2) was obtained by Kolaitis, Prömel, and Rothschild [KPR85, KPR87], who showed that almost every K_{p+1}-free graph is p-colorable. This result was further extended by Prömel and Steger [PS92b] (see also [HPS93]) from cliques to such graphs L, which contain a *color-critical edge*, i.e., an edge $e \in E(L)$ such that $\chi(L-e) < \chi(L)$ with $L - e = (V(L), E(L) \setminus \{e\})$. The result of Prömel and Steger states that for graphs L with $\chi(L) = p + 1 \geq 3$ almost every L-free graph is p-colorable if and only if L contains a color-critical edge, which was conjectured earlier by Simonovits [Sim09].

Recently, Balogh, Bollobás, and Simonovits [BBS04] showed a sharper version of (1.3):

$$|\mathrm{Forb}(n, L)| \leq 2^{(1-\frac{1}{p})\binom{n}{2}+O(n^{2-\gamma})},$$

where $p = \chi(L) - 1$ and $\gamma = \gamma(L) > 0$ is some constant (best possible) depending on L (see also [BBS09, BBS] for more structural results by the same authors).

Using the hypergraph regularity lemma, Nagle, Rödl and Schacht [NR01, NRS06b] generalized (1.3) to k-uniform hypergraphs, i.e,

$$|\mathrm{Forb}(n, L)| \leq 2^{\mathrm{ex}(n,L)+o(n^k)} \tag{1.4}$$

for arbitrary k-uniform hypergraphs L.

In Chapter 4 we will sharpen the bound (1.4) on $|\mathrm{Forb}(n, F)|$ in the special case, when F is the 3-uniform hypergraph of the Fano plane.

Let $\mathcal{B}_n$ be the class of all labeled 2-colorable (or bipartite) hypergraphs on n vertices, where we say that a hypergraph is bipartite if there exists a bipartition of its vertex set such that every hyperedge intersects both sets. The hypergraph F of the Fano plane, or the Fano plane for short, is defined to be the (unique) 3-uniform hypergraph on 7 vertices with 7 hyperedges such that any two of them intersect in exactly one point (alternatively it arises naturally from the smallest projective geometry). The Fano plane is not bipartite but becomes so on deleting an arbitrary hyperedge. Therefore, the class

$\mathcal{B}_n$ is contained in $\mathrm{Forb}(n, F)$. On the other hand, we prove that almost every Fano-free hypergraph is 2-colorable.

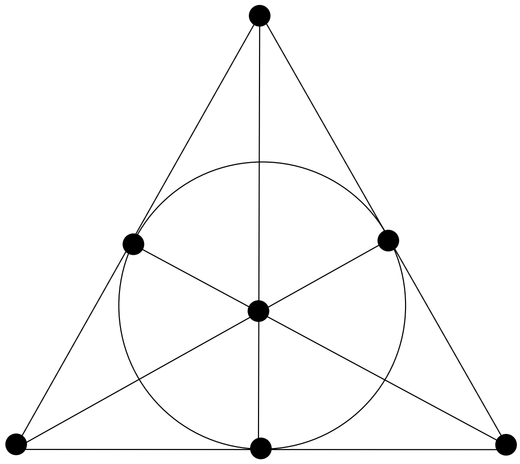

Figure 1.3: The 3-uniform hypergraph of the Fano plane

Theorem 1.7. *Let F be the 3-uniform hypergraph of the Fano plane. There exist a real $c > 0$ and an integer n_0, such that for every $n \geq n_0$ we have*

$$|\mathrm{Forb}(n, F)| \leq |\mathcal{B}_n|(1 + 2^{-cn^2}). \tag{1.5}$$

The above theorem can be seen as a first attempt to derive results for hypergraphs in the spirit of Kolaitis, Prömel and Rothschild [KPR87] and Prömel and Steger [PS92b]. We will prove Theorem 1.7 using techniques developed by Balogh, Bollobás and Simonovits in [BBS04]. In fact, with these methods, one could also reprove the above mentioned theorems for graphs containing color-critical edges [BS08].

More recently, Balogh and Mubayi [BMa, BMb], using a version of strong hypergraph regularity lemma of Frankl and Rödl [FR02], studied $\mathrm{Forb}(n, L)$, when L is the generalized triangle T_3 (which will be studied to some extent in Chapter 5) or L is the hypergraph $F_{2,3}$, where $E(T_3) := \{abc, abd, cde\}$ and $E(F_{2,3}) := \{abc, abd, abe, cde\}$. Those authors showed that almost all members from these classes look like subgraphs of their (unique) extremal hypergraphs.

An algorithmic application

One of the classical problems in complexity theory is to decide whether a given k-uniform hypergraph is 2-colorable (or *bipartite*). While for bipartite graphs a 2-coloring can be found in linear time, it was shown by Lovász [Lov73] that the problem becomes NP-complete for k-uniform hypergraphs and $k \geq 3$. Moreover, Guruswami, Håstad and Sudan [GHS02] proved that it is NP-hard to color bipartite, k-uniform hypergraphs with a constant number of colors for $k \geq 4$. Further it was shown by Dinur, Regev and Smyth [DRS05] that this problem remains inapproximable by a constant for 3-uniform hypergraphs. On the other hand, Krivelevich, Nathaniel and Sudakov [KNS01] found a polynomial time algorithm which colors 3-uniform bipartite hypergraphs using $O(n^{1/5}\log^c n)$ colors. Another positive result is due to Chen and Frieze [CF96]. Those authors studied colorings of so-called α-dense bipartite 3-uniform hypergraphs, where a 3-uniform hypergraph is α-dense if the joint degree of any two vertices is at least αn. A randomized algorithm that can color H in $n^{O(1/\alpha)}$ time was found [CF96].

Developing ideas that led to Theorem 1.7, we present in Section 4.3 an algorithm that colors a hypergraph chosen uniformly at random from the family of all labeled 3-uniform bipartite hypergraphs on n vertices in $O(n^5\log^2 n)$ expected time. Indeed, we prove a slightly more general result for the class of Fano-free hypergraphs, see Theorem 1.8. Before we state it precisely we review related results for graphs.

In 1984 Wilf [Wil84] noted, using a simple counting argument, that one can decide in constant expected time, whether a graph is ℓ-colorable. Few years later Turner [Tur88] gave an $O(|V|+|E|\log\ell)$ algorithm for optimally coloring almost all ℓ-colorable graphs. This result was further expanded by Dyer and Frieze [DF89] who developed an algorithm which colors every ℓ-colorable graph on n vertices properly (with ℓ colors) in $O(n^2)$ expected time.

Another line of research is the study of monotone properties of the type $\mathrm{Forb}(n, L)$ for a fixed graph L. Prömel and Steger [PS92c] gave an algorithm that colors properly (regardless of the value $\chi(G)$) a randomly chosen member G from $\mathrm{Forb}(n, K_{\ell+1})$, i.e., the class of all labeled $K_{\ell+1}$-free graphs, in $O(n^2)$ expected time. This is clearly a generalization of the result of Dyer and Frieze in the light of the well known result of Kolaitis, Prömel and Rothschild [KPR87] that almost all $K_{\ell+1}$-free graphs are ℓ-colorable.

Motivated by the aforementioned result of Prömel and Steger [PS92c], we present an application of Theorem 1.7 which can be stated as follows.

Theorem 1.8. *There is an algorithm with average running time $O(n^5\log^2 n)$ which colors every member from* $\mathrm{Forb}(n, F)$ *properly, where F is the hypergraph of the Fano plane.*

Together with (1.5) we immediately derive in a similar manner to [PS92c] that one can color a 3-uniform hypergraph chosen uniformly at random from $\mathcal{B}_n$ in polynomial expected time.

Corollary 1.9. *There is an algorithm with average running time $O(n^5\log^2 n)$ which finds a bipartition of every member from $\mathcal{B}_n$.*

All results of the above section were discovered in collaboration with Mathias Schacht [PS09a, PS09b, PS].

1.2.3 Restricted hyperedge coloring problems

Historically, the following problem posed by Erdős and Rothschild [Erd74] was at the beginning of the investigations concerning the class $\mathrm{Forb}(n, F)$. What is the maximum number of 2-edge colorings a graph on n vertices can admit without having a monochromatic copy of F? Here we mean by an r-coloring a function c from the edge set of H into $\{1, \ldots, r\}$, i.e. not necessarily a proper coloring. Let us denote this function by $c_{2,F}(n)$ and the maximum number of such colorings for a graph G by $c_{2,F}(G)$, where a formal definition of $c_{r,F}(G)$ is

$$\left| \{c \mid c \colon E(G) \to [r], \text{ with } c^{-1}(i) \not\supseteq F \; \forall i \in [r]\} \right| .$$

We will sometimes call these colorings F-free.

In the graph case, when $F = K_\ell$ is a graph clique Yuster [Yus96] (for $\ell = 3$, and $n \geq 6$) and Alon et al. [ABKS04] (for arbitrary ℓ and large n) showed, that $c_{2,K_\ell}(n) = 2^{\mathrm{ex}(n,K_\ell)}$, which was conjectured by Erdős and Rothschild (see [Erd74]). Moreover, Alon, Balogh, Keevash, and Sudakov showed that $c_{3,K_\ell}(n) = 3^{\mathrm{ex}(n,K_\ell)}$ and in both cases $r = 2, 3$ and n large we have

$$c_{r,K_\ell}(H) = c_{r,K_\ell}(n) = r^{\mathrm{ex}(n,K_\ell)}$$

only when H is the $(\ell - 1)$-partite Turán graph. In fact, it was shown in [ABKS04] that the same result holds for ℓ-chromatic graphs which contain a color-critical edge. Furthermore, it was observed in [ABKS04] that $c_{r,K_\ell}(n) \gg r^{\mathrm{ex}(n,K_\ell)}$ for $r \geq 4$. Recently, Pikhurko and Yilma [PY] determined the graphs that yield $c_{4,K_3}(n)$ and $c_{4,K_4}(n)$.

In a series of papers [LPRS09, LPS, LP] we studied $c_{r,F}(n)$ for k-uniform hypergraphs and in Chapter 5 we present the extensions of the results of Alon et al.

For k-uniform hypergraphs F and H and an integer r let $c_{r,F}(H)$ denote the number of r-colorings of the set of hyperedges of H with no monochromatic copy of F and let $c_{r,F}(n) = \max_{H \in \mathcal{H}_n} c_{r,F}(H)$, where the maximum runs over all k-uniform hypergraphs on n vertices.

Clearly, every edge coloring of any extremal hypergraph H for F contains no monochromatic copy of F and, consequently,

$$c_{r,F}(n) \geq r^{\mathrm{ex}(n,F)}$$

for all $r \geq 2$. On the other hand, recall that $\mathrm{Forb}(n, F)$ denotes the family of all labeled hypergraphs on n vertices which contain no copy of F. Since every 2-coloring of the hyperedges of a hypergraph H, which contains no monochromatic copy of F, gives rise to a member of $\mathrm{Forb}(n, F)$, e.g., consider always the subhypergraph in one of the two colors, we have

$$c_{2,F}(n) \leq |\mathrm{Forb}(n, F)| .$$

Thus, from our discussion, see (1.4), in the previous section we obtain:

$$2^{\mathrm{ex}(n,F)} \leq c_{2,F}(n) \leq 2^{\mathrm{ex}(n,F)+o(n^k)} . \tag{1.6}$$

A structural result

Our first result of Chapter 5 is related to (1.6). It gives an upper bound on $c_{r,F}(n)$ for $r = 2, 3$. Moreover, we show a further leading result stating that for a natural class of hypergraphs F, any hypergraph H for which many F-free colorings exist must disclose a special structure. This natural class of hypergraphs is captured by the notion of s-stability, introduced by Pikhurko in [Pik05].

Definition 1.10 (s-stability). *Let F be a k-uniform hypergraph with positive Turán density π_F, which is defined as*

$$\lim_{n \longrightarrow \infty} \frac{\mathrm{ex}(n, F)}{\binom{n}{k}}.$$

Call F s-stable, if for every $\varepsilon > 0$ there exists an $\omega > 0$ and an integer n_0 such that for arbitrary F-free k-uniform hypergraphs $H_1, \dots, H_{s+1}$ each with the same number of vertices $n \geq n_0$ and each having at least $\pi_F\binom{n}{k} - \omega n^k$ hyperedges, there are two which are ε-close. By ε-close we mean that one can delete or add at most εn^k hyperedges from the first hypergraph to obtain an isomorphic copy of the second hypergraph.

Simonovits [Sim68] and independently Erdős [Erd68] showed that graphs are 1-stable. However, for hypergraphs such a result is not known and it is believed that in general it even fails for $k \geq 3$ and k-uniform hypergraphs. There are however hypergraphs, for which the extremal hypergraphs are known and moreover it is known that they are stable [FF89, DCF00, KM04, FS05, FPS05, KS05b, KS05a, Pik05, FPS06, Mub06, MP07, FMP08, Pik08, Pik].

With the definition of stability at hand we can state the following result.

Theorem 1.11. *Let $k, s \in \mathbb{N}$, $k \geq 2$ and $r = 2$ or 3. Let F be a k-uniform hypergraph, such that its Turán density $\pi_F > 0$.*

Then, for every $\varepsilon > 0$ there exists an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ it is

$$c_{r,F}(n) \leq r^{\mathrm{ex}(n,F)+\varepsilon n^k}. \tag{1.7}$$

Furthermore, suppose that F is s-stable. Then, among any $(s+1)$ k-uniform hypergraphs $H_1, \dots, H_{s+1}$ on $n \geq n_0$ vertices that satisfy $c_{r,F}(H_i) \geq r^{\mathrm{ex}(n,F)}$ for every $i \in [s+1]$, there exist two which are ε-close.

Note that for the general upper bound (1.7) on the number $c_{r,F}(n)$, s-stability is not required. This upper bound (1.7) also holds for those hypergraphs F with $\pi_F = 0$ and an arbitrary fixed number of colors. This is a triviality due to the assumption $\pi_F = 0$.

Exact results

In a similar spirit to [ABKS04] we utilize Theorem 1.11 and determine $c_{r,F}(n)$ for $r = 2, 3$ and n large enough exactly for various (families of) hypergraphs for which extremal results are known. These hypergraphs will be the Fano plane, introduced earlier, the 3- and 4-uniform generalized triangles, expanded complete graphs and Fan(k)-hypergraphs, which we introduce properly below. The common feature is that they are all 1-stable (and they possess unique extremal hypergraphs).

The theorems concerning $c_{r,F}(n)$ all assert that for $r = 2, 3$ and n large enough the only hypergraphs attaining $c_{r,F}(n)$ are extremal hypergraphs and for $r \geq 4$ one has $c_{r,F}(n) \gg r^{\mathrm{ex}(n,F)}$.

Fano plane The exact result for the hypergraph of the Fano plane reads as follows.

Theorem 1.12. *Let F be the 3-uniform hypergraph of the Fano plane and $r = 2$ or $r = 3$. There exists an integer n_r, such that for every hypergraph H on $n \geq n_r$ vertices we have*

$$c_{r,F}(H) \leq r^{\mathrm{ex}(n,F)}.$$

Moreover, the only hypergraph H on n vertices with $c_{r,F}(H) = r^{\mathrm{ex}(n,F)}$ is the extremal hypergraph for F, i.e., H is isomorphic to B_n the balanced, complete, bipartite hypergraph on n vertices.

Generalized triangles For an integer $k \geq 2$, define the generalized triangle T_k as follows. This k-uniform hypergraph $T_k = (V, E)$ has the vertex set $V = [2k-1]$ and its set E of three hyperedges is given by

$$E = \{\{1, \ldots, k\}, \{1, \ldots, k-1, k+1\}, \{k, k+1, \ldots, 2k-1\}\}.$$

Thus, the first two hyperedges have $(k-1)$ common vertices, while the third hyperedge

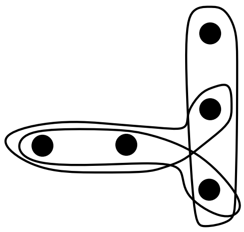

Figure 1.4: T_3, the 3-uniform generalized triangle

contains the symmetric difference of the first two, and intersects each of these in precisely one vertex. Clearly, if $k = 2$, then T_2 is a graph triangle K_3.

The following hypergraph $\mathcal{T}_k^{(k)}(n)$, called the Turán hypergraph, does not contain any copy of T_k and is defined as follows. $\mathcal{T}_k^{(k)}(n)$ is the complete k-partite k-uniform hypergraph with vertex classes as equal as possible. For $k = 3$ it was shown by Frankl and Füredi [FF83] and for $k = 4$ by Pikhurko [Pik08], that the Turán hypergraph $\mathcal{T}_k^{(k)}(n)$ is the unique extremal T_k-free hypergraph for n sufficiently large. Moreover, Keevash and Mubayi [KM04] (case $k = 3$) and Pikhurko [Pik08] (case $k = 4$) showed that T_k is 1-stable. For $k = 5, 6$, the hypergraph $\mathcal{T}_k^{(k)}(n)$ is not extremal for T_k anymore [FF89]. Furthermore one can also extend the constructions from [FF89] to show that for $k \geq 7$, $\mathcal{T}_k^{(k)}(n)$ is not extremal for T_k as well.

For T_3 and T_4 we have the following result.

Theorem 1.13. *Let $k = 3$ or 4 and $r = 2$ or 3. There exists an integer $n_{r,k}$, such that*

$$c_{r,T_k}(H) \leq r^{\mathrm{ex}(n,T_k)}$$

for any k-uniform hypergraph H on $n \geq n_{r,k}$ vertices. Moreover, if $c_{r,T_k}(H) = r^{\mathrm{ex}(n,T_k)}$ then H is isomorphic to the Turán hypergraph $\mathcal{T}_k^{(k)}(n)$.

Expanded complete graphs and Fan(k)-hypergraphs Before we define these special hypergraphs, we introduce a generalization of $\mathcal{T}_k^{(k)}(n)$, being the k-uniform hypergraph $\mathcal{T}_\ell^{(k)}(n)$, which we will refer to as *Turán* hypergraph, and for $k = 2$, $\mathcal{T}_\ell^{(k)}(n)$ will be the usual Turán graph. Partition the vertex set $[n]$ into ℓ mutually disjoint subsets $V_1, \ldots, V_\ell$ of sizes as equal as possible, i.e., they differ in size by at most 1. Then, consider as hyperedges all k-element subsets of $[n]$ that intersect every partition class V_i, $i \in [\ell]$, in at most one vertex. It is easy to check that the Turán hypergraph $\mathcal{T}_\ell^{(k)}(n)$ contains the maximum possible number of hyperedges with the property that every hyperedge intersects every class V_i, $i \in [\ell]$, in at most one vertex, and is unique up to isomorphism. We have the following bounds on its number of hyperedges

$$\binom{\ell}{k} \cdot \left\lfloor \frac{n}{\ell} \right\rfloor^k \leq e(\mathcal{T}_\ell^{(k)}(n)) \leq \binom{\ell}{k} \cdot \left\lceil \frac{n}{\ell} \right\rceil^k, \tag{1.8}$$

and the following bounds on the minimum degree $\delta(\mathcal{T}_\ell^{(k)}(n))$ of $\mathcal{T}_\ell^{(k)}(n)$ hold:

$$\binom{\ell-1}{k-1} \cdot \left\lfloor \frac{n}{\ell} \right\rfloor^{k-1} \leq \delta(\mathcal{T}_\ell^{(k)}(n)) \leq \binom{\ell-1}{k-1} \cdot \left\lceil \frac{n}{\ell} \right\rceil^{k-1}, \tag{1.9}$$

as every class $|V_i|$ has size at least $\lfloor n/\ell \rfloor$ and at most $\lceil n/\ell \rceil$.

For integers $\ell, k \geq 2$, we define the so-called *expanded complete graph* $H_{\ell+1}^k$, sometimes called expanded clique, to be the k-uniform hypergraph obtained as follows. We take $\binom{\ell+1}{2}$ edges of the complete graph $K_{\ell+1}$ on the vertices $v_1, \ldots, v_{\ell+1}$, called the *core* of the hypergraph $H_{\ell+1}^k$, and we enlarge every edge by a set of $(k-2)$ new vertices. Thus, the vertex set of the hypergraph $H_{\ell+1}^k$ has size $(\ell+1) + \binom{\ell+1}{2} \cdot (k-2)$ and it contains $\binom{\ell+1}{2}$ hyperedges. Clearly we have inclusions $H_{\ell+1}^k \supset H_\ell^k$.

Similarly, for integers ℓ, $k \geq 2$, $\ell \geq k-1$, we define the *Fan(k)-hypergraph* $F_{\ell+1}^k$ to be the k-uniform hypergraph, which contains $(\ell+1)$ vertices $v_1, \ldots, v_{\ell+1}$ called the *core* of $F_{\ell+1}^k$. Moreover, k vertices of this core form a hyperedge, the *core-hyperedge,* say these are the vertices $v_1, \ldots, v_k$, and then for each $\{i,j\} \in [\{1, \ldots, \ell+1\}]^2 \setminus [\{1, \ldots, k\}]^2$ the two-element set $\{v_i, v_j\}$ is enlarged by a set of $(k-2)$ new vertices. Hence, the vertex set of $F_{\ell+1}^k$ has size $(\ell+1) + (\binom{\ell+1}{2} - \binom{k}{2}) \cdot (k-2)$ and it contains $1 + \binom{\ell+1}{2} - \binom{k}{2}$ hyperedges. Note that F_k^k contains exactly one hyperedge. These families have been studied by Mubayi and Pikhurko [Mub06, Pik05, MP07], where it is shown that for large n and $\ell \geq k$ the unique extremal hypergraph for $H_{\ell+1}^k$ [Pik05] and $F_{\ell+1}^k$ [MP07] as well is the Turán hypergraph $\mathcal{T}_\ell^{(k)}(n)$. Moreover, for the hypergraphs $H_{\ell+1}^k$ and $F_{\ell+1}^k$, it is known that they are 1-stable [Pik05, MP07].

Theorem 1.14. *Let $F = H_{\ell+1}^k$ be the k-uniform, expanded, complete graph or $F = F_{\ell+1}^k$ the Fan(k)-hypergraph, both with core of size $(\ell+1)$, where $2 \leq k \leq \ell$. Let $r = 2$ or*

$r = 3$.

Then, there exists a positive integer $n_r(F)$, such that for every k-uniform hypergraph H on $n \geq n_r(F)$ vertices it is

$$c_{r,F}(H) \leq r^{\mathrm{ex}(n,F)}.$$

Moreover, for $r = 2$ or $r = 3$, and n sufficiently large, the only hypergraph H on n vertices with $c_{r,F}(H) = r^{\mathrm{ex}(n,F)}$, is the extremal hypergraph for F, i.e., H is isomorphic to $\mathcal{T}_\ell^{(k)}(n)$, the k-uniform Turán hypergraph on n vertices with ℓ classes.

Note that for $k = 2$ we have $H^k_{\ell+1} = F^k_{\ell+1} = K_{\ell+1}$ and $\mathcal{T}_\ell^{(k)}(n)$ is the usual Turán graph. The case $k = 2$ is the result of Alon et al.[ABKS04].

Further tools used in the study of this problem are stability theorems, and this approach resembles stability method used quite recently extensively to determine hypergraph extremal numbers. However to prove an exact result from an approximate one is sometimes far more difficult than the corresponding result about the extremal number. The methods have some common features with those from Chapter 4.

More than 3 colors and a general upper bound for $r \geq 4$

The following results show that, similarly as in the case of graph cliques [ABKS04], Theorems 1.12, 1.13 and 1.14 do not extend to more than 3 colors.

Theorem 1.15. *Let $r > 3$ and F be either the Fano plane, T_3, T_4, $H^k_{\ell+1}$ or $F^k_{\ell+1}$ with $\ell \geq k \geq 2$. Then for sufficiently large n we have*

$$c_{r,F}(n) \gg r^{\mathrm{ex}(n,F)}. \tag{1.10}$$

We also give an upper bound on $c_{r,F}(n)$ for any fixed, k-uniform hypergraph F with positive Turán density and for any fixed integer $r \geq 4$.

Theorem 1.16. *Let F be any k-uniform hypergraph F with Turán density $\pi_F > 0$. For a fixed integer $r \geq 4$ it is*

$$c_{r,F}(n) \leq (\pi_F \cdot r)^{\binom{n}{k}+o(n^k)} \quad \text{if} \quad \pi_F \cdot r \geq e,$$

and

$$c_{r,F}(n) \leq e^{(r/e)(\pi_F+o(1))\binom{n}{k}} \quad \text{if} \quad \pi_F \cdot r < e.$$

Results in this chapter were obtained in joint work with Hanno Lefmann, Vojtěch Rödl and Mathias Schacht [LPRS09, LPS, LP].

2 Tools

2.1 Notation and preliminaries

2.1.1 Basics

By $\mathbb{R}$ we denote the set of real numbers. For real constants α, β, and a non-negative constant ξ we sometimes write $\alpha = \beta \pm \xi$ if $\beta - \xi \leq \alpha \leq \beta + \xi$. By $\mathbb{N}$ we denote the set of natural numbers without zero. For $n \in \mathbb{N}$ we set $[n] := \{1, \dots, n\}$, $(n)_k$ denotes the falling factorial $\prod_{j=0}^{k-1}(n-j)$, with $(n)_n = n!$. The binomial coefficients are $\binom{n}{k} := (n)_k/k!$ and we use a well known estimate via the entropy function $h(x) := -x \log x - (1-x)\log(1-x)$ for $x \in (0,1)$:

$$\binom{n}{\alpha n} \leq 2^{h(\alpha)n} \tag{2.1}$$

for $\alpha \in (0,1)$ and large n. Note also that $h(x)$ goes to 0 as x tends to 0. Here and anywhere else we mean by log the logarithm to the base 2, $\log_2$, while ln stands for the natural logarithm.

For a set V and an integer $k \geq 1$, let $[V]^k$ be the set of all k-element subsets of V, called k-sets for short. We may drop one pair of brackets and write $[n]^k$ instead of $[[n]]^k$. We define $[V]^{\leq k} := \bigcup_{i \in [k]} [V]^i$, i.e. $[V]^{\leq k}$ consists of all nonempty subsets of V of cardinality at most k. By V^k we will denote the Cartesian product of V, that is the set of (ordered) k-tuples $(v_1, \dots, v_k)$ with $v_i \in V$ for every $i \in [k]$.

For functions $f, g \colon \mathbb{N} \to \mathbb{R}_{\geq 0}$ we write $f = o(g)$ if $|f(n)| \leq c(n)|g(n)|$ for large n, with $c(n)$ tending to 0 as n goes to infinity, and simply $f = o(1)$ if $\lim_{n\to\infty} f(n) = 0$; $f = O(g)$ if the quotient $|f(n)/g(n)|$ is bounded as n tends to infinity, and $f = \Omega(g)$ if $g = O(f)$. If both, $f = O(g)$ and $f = \Omega(g)$ holds, we write $f = \Theta(g)$.

For vectors $x, y \in \mathbb{R}^n$ the Cauchy-Schwarz inequality states:

$$\left|\sum_{i=1}^{n} x_i \cdot y_i\right|^2 \leq \sum_{j=1}^{n} x_j^2 \cdot \sum_{k=1}^{n} y_k^2. \tag{2.2}$$

2.1.2 Graphs

A graph $G = (V, E)$ is a tuple of two sets V and E with $E \subseteq [V]^2$. We call elements of V vertices and elements of E edges, thus V is the vertex set of G and E is the edge set of it. For $e \in E$ we might explicitly name the vertices e is incident with, say these are $x, y \in V$, thus we write $e = \{x, y\}$. In order to further simplify the notation it will be convenient to sometimes drop the brackets and write xy for e (and thus for $\{x, y\}$).

Given another graph $G' = (V', E')$, we say G' is a subgraph of G if there exists an injective map $\varphi\colon V' \to V$ such that whenever $e' \in E'$ it follows $\varphi(e') \in E$, where for a subset $S \subseteq V$ we define $\varphi(S) := \{\varphi(s)\colon s \in S\}$. We call φ, and $(\varphi(V'), \varphi(E'))$, a labeled copy of G' in G. If G' is not a subgraph of G we say that G is G'-free. For a subset $U \subseteq V$ we denote by $G[U] := (U, E \cap [U]^2)$ the subgraph of G *induced* on U. The edge set of $G[U]$ is denoted by $E_G(U)$, and we set $e_G(U) := |E_G(U)|$. For two disjoint subsets $U, W \subseteq V$ we write $G[U, W] = (U, W)_G$ which is the bipartite subgraph of G whose edges are exactly those edges of G that intersect both sets U and W. By the density of (U, W) we mean the quantity:

$$d_G(U, W) := \frac{e_G(U, W)}{|U| \cdot |W|}.$$

For each vertex $v \in V$ we set $N_G(v) := \{u \in V\colon uv \in E\}$ to be the neighborhood of v in G and $\deg(v) := |N_G(V)|$ is said to be its degree. We further set

$$\delta(G) := \min\{\deg(v)\colon v \in V\} \quad \text{and} \quad \Delta(G) := \max\{\deg(v)\colon v \in V\}$$

for the minimum and maximum degrees of G, respectively.

A k-coloring of G is a function $c\colon V \to [k]$. The coloring c is called proper if whenever $xy \in E$ we have $c(x) \neq c(y)$. The chromatic number of G, $\chi(G)$, is the smallest k for which a proper k-coloring exists. For $\ell \geq \chi(G)$ we say G is ℓ-colorable.

Some special graphs we are interested in are the complete graph $K_n = ([n], [n]^2)$, a path $P_n = ([n], E_{P_n})$ on n vertices, where $E = \{\{i, i+1\} : 1 \leq i < n\}$ and a cycle $C_n = ([n], E_{P_n} \cup \{1, n\})$. The graph Q_k of the k-dimensional cube has the vertex set $\{0, 1\}^k$, the binary vectors of length k, and two vertices are adjacent if they differ in only one coordinate. A matching $M \subseteq E$ of a graph $G = (V, E)$ is a set of edges such that no two of them have a common vertex.

A graph G is bipartite, if there exists a partition of its vertex set into two sets V_1 and V_2 such that no edge lies within V_1 or V_2. We call V_1 and V_2 (color) classes of $V(G)$, as a bipartite graph is 2-colorable. More generally, for an ℓ-*partite* graph $G = (V, E)$ there exists a partition $V = V_1 \dot\cup \ldots \dot\cup V_\ell$ such that whenever $xy \in E$ there exist $i, j \in [\ell], i \neq j$ with $xy \in E(V_i, V_j)$, in other words no edge lies within a single class. An example is $K_\ell(r_1, \ldots, r_\ell)$, the complete ℓ-partite graph with class sizes $r_1, \ldots, r_\ell$, i.e. all edges between any two classes are present in that graph.

2.1.3 Hypergraphs

A k-uniform hypergraph $H^{(k)}$ is a tuple (V, E) where V is the set of vertices and E is the set of hyperedges with $E \subseteq [V]^k$. With this notation, a graph is a 2-uniform hypergraph. We will sometimes call hyperedges simply edges and omit k when it is clear from the context, further it is often convenient to identify hypergraphs with the sets of their hyperedges. Similarly to graphs, we write $V(H)$ and $E(H) := H$ for the vertex and edge set respectively. Given a hypergraph $H' = (V', E')$, we say H' is a subhypergraph of H if there exists an injective map $\varphi\colon V' \to V$ such that whenever

$e' \in E'$ it follows $\varphi(e') \in E$, where for a subset $S \subseteq V$ we define $\varphi(S) := \{\varphi(s) \colon s \in S\}$. We call φ, and $(\varphi(V'), \varphi(E'))$, a labeled copy of H' in H. We also sometimes refer to H as a *superhypergraph* of H'. We similarly set for $U, W \subseteq V \colon H[U] = (U, E \cap [U]^k)$, $E_H(U) := E \cap [U]^k$ and $E(U, W) := \{f \mid f \cap U \neq \varnothing \neq f \cap W\,, f \in E\}$. Also, we write $e_H(U)$ for $|E_H(U)|$, $e_H(U, W)$ for $|E_H(U, W)|$ etc. More generally, for a given k-uniform hypergraph $H = (V, E)$ and k mutually disjoint subsets of its vertex set $V_1, \ldots, V_k \subseteq V$ we write $E_H(V_1, \ldots, V_k)$ for the set of those hyperedges from H that intersect each V_i in exactly one vertex and $e_H(V_1, \ldots, V_k) := |E_H(V_1, \ldots, V_k)|$. We define

$$d_H(V_1, \ldots, V_k) := \frac{e_H(V_1, \ldots, V_k)}{|V_1| \cdot \ldots \cdot |V_k|}$$

as the density of the k-tuple $(V_1, \ldots, V_k)$.

For a vertex $v \in V(H)$ we set

$$L_H(v) := (V \setminus \{v\}, E_v), \quad \text{where} \quad E_v := \{e \setminus \{v\} \colon v \in e \in E(H)\},$$

and we call it the link hypergraph, or simply the link, of v, and the degree of v is $\deg(v) := |L_H(v)|$. Note that $L_H(v)$ is then a $(k-1)$-uniform hypergraph. The notations $\delta(H), \Delta(H)$ stand again for the minimum and maximum degree, respectively:

$$\delta(H) := \min\{\deg(v) \colon v \in V\} \quad \text{and} \quad \Delta(H) := \max\{\deg(v) \colon v \in V\}.$$

Some special hypergraphs that we will be interested in are $K_\ell^{(k)} = ([\ell], [\ell]^k)$, the complete k-uniform hypergraph on ℓ vertices, sometimes called k-uniform clique of order ℓ; the hypergraph of the Fano plane, which is the unique 3-uniform hypergraph on 7 vertices with 7 hyperedges such that any two of them intersect in exactly one vertex; $H_{\ell+1}^k$, which is obtained by enlarging every edge of $K_{\ell+1}^{(2)}$ by $(k-2)$ new vertices (thus it becomes k-uniform). More generally, the last two hypergraphs are the so-called *linear* hypergraphs (also known as simple), i.e. any two hyperedges of a linear hypergraph intersect in at most one vertex.

We say two (k-uniform) hypergraphs $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ are isomorphic, if there exists a bijection $\varphi \colon V_1 \to V_2$ such that for every $e \in [V_1]^k$ we have $e \in E_1$ if and only if $\varphi(e) \in E_2$. Moreover, two hypergraphs H_1 and H_2 on the same number of vertices, say n, are called ε-close for some $\varepsilon > 0$ if there exists a bijection $\varphi \colon V_1 \to V_2$ such that

$$|\varphi(E_1) \Delta E_2| \leq \varepsilon n^k,$$

where Δ denotes the symmetric difference. In other words, by deleting and/or adding at most εn^k hyperedges to/from H_1 we can obtain a hypergraph isomorphic to H_2 (and vice versa).

A labeled copy of some k-uniform hypergraph $F = (V(F), E(F))$ in another k-uniform hypergraph $H = (V, E)$ is an injective function $\varphi \colon V(F) \to V$ such that whenever $f \in E(F)$ it follows $\varphi(f) \in E$. We denote by $N_F(H)$ the number of labeled copies of F in H. And for pairwise disjoint sets $U_1, \ldots, U_\ell \subseteq V(H)$ we write $N_F(U_1, \ldots, U_\ell)$ for the number of partite-isomorphic, copies of F in H, i.e., the number of ℓ-tuples $(h_1, \ldots, h_\ell)$ with $h_1 \in U_1, \ldots, h_\ell \in U_\ell$ such that $\{h_{i_1}, \ldots, h_{i_k}\}$ is an edge in H if $\{i_1, \ldots, i_k\}$ is an edge in F.

Sometimes we will use the notion of an induced copy of F in H. An induced copy is an injective function $\varphi\colon V(F) \to V$ such that a k-element set $f \in E(F)$ if and only if $\varphi(f) \in E$. We denote by $N^*_F(H)$ the number of induced copies of F in H. Similarly to $N_F(U_1, \ldots, U_\ell)$, let $U_1, \ldots, U_\ell \subseteq V(H)$ be pairwise disjoint sets. We write $N^*_F(U_1, \ldots, U_\ell)$ for the number of partite-isomorphic, copies of F in H, i.e., the number of ℓ-tuples $(h_1, \ldots, h_\ell)$ with $h_1 \in U_1, \ldots, h_\ell \in U_\ell$ such that $\{h_{i_1}, \ldots, h_{i_k}\}$ is an edge in H if and only if $\{i_1, \ldots, i_k\}$ is an edge in F.

Another fruitful and useful concept is that of homomorphisms. For two k-uniform hypergraphs F and H, we say that a function $\varphi\colon V(F) \to V(H)$ is a homomorphism from F to H if whenever $f \in E(F)$ there always holds $\varphi(f) \in E(H)$. Thus, k-partite k-uniform hypergraphs are homomorphic to a single hyperedge. where a k-partite k-uniform hypergraph admits a partition of its vertex set into k classes such that every hyperedge intersects every class in *exactly* one vertex. More generally, we will define partite hypergraphs in two slightly different ways. When $\ell < k$ then we say that a k-uniform hypergraph is ℓ-partite if it admits a partition of its vertex set into ℓ classes such that no hyperedge lies completely within any class, while for the case $\ell \geq k$ we say that a k-uniform hypergraph is ℓ-partite if it admits a partition of its vertex set into ℓ classes such that every hyperedge intersects every class in *at most one* vertex, and we will refer to such hyperedges as crossing. As for notation, for $k \leq \ell$, we write $K^{(k)}_\ell(V_1, \ldots, V_\ell)$ to denote the complete ℓ-partite k-uniform hypergraph with partition classes $V_1, \ldots, V_\ell$, where every crossing hyperedge is present.

2.1.4 Extremal problems for hypergraphs

One of the most extensively studied functions in extremal graph theory is the extremal (or sometimes called Turán) function $\mathrm{ex}(n, F)$ for $n \in \mathbb{N}$ and a graph F. It denotes the maximum possible number of edges a graph on n vertices can have without containing a copy of F as a subgraph. Those F-free graphs on n vertices attaining $\mathrm{ex}(n, F)$ edges are called extremal graphs. Similarly, $\mathrm{ex}(n, \mathcal{F})$ stands for the maximum number of edges a graph on n vertices can have without containing a copy of some F from the family $\mathcal{F}$.

A well known theorem of Turán [Tur41] establishes that for $r \geq 2$ the K_{r+1}-extremal graph on n vertices is unique. It is the balanced (balanced means that the sizes of the vertex classes differ by at most one) complete r-partite graph on n vertices, denoted by $T_r(n)$ and this graph is called the Turán graph. The result $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$ and that extremal graph is $K_2(\lfloor n/2 \rfloor, \lceil n/2 \rceil)$ was shown by Mantel [Man07].

A theorem due to Erdős and Stone [ES46] asserts that $\mathrm{ex}(n, K_\ell(s, \ldots, s)) = \mathrm{ex}(n, K_\ell) + o(n^2)$ and its generalization due to Erdős and Simonovits [ES66] to general ℓ-chromatic graphs states that

$$\mathrm{ex}(n, F) = \mathrm{ex}(n, K_{\chi(F)}) + o(n^2).$$

This already gives a good characterization of the so-called non-degenerate case: the number of the edges of extremal graph(s) is quadratic in the number of its vertices. We define the Turán density π_F for any graph F to be

$$\pi_F := \lim_{n \longrightarrow \infty} \frac{\mathrm{ex}(n,F)}{\binom{n}{2}},$$

which is well defined, as $\mathrm{ex}(n,F)/\binom{n}{2}$ is monotone decreasing function as was noted by Katona, Nemetz and Simonovits [KNS64]. In the case when F is bipartite it is a consequence from the result of Kövari, Turán and Sós [KST54] that $\mathrm{ex}(n,F) = o(n^2)$ and therefore $\pi_F = 0$.

For hypergraphs, extremal numbers and extremal hypergraphs are defined verbatim as for graphs, but there is no unified classification of even the Turán density. A well known open question is π_F for $F = K_4^{(3)}$ [Tur41], where again, for a fixed given k-uniform hypergraph F we set

$$\pi_F := \lim_{n \longrightarrow \infty} \frac{\mathrm{ex}(n,F)}{\binom{n}{k}},$$

which is well defined and exists [KNS64].

Said all that, there are however hypergraphs for which Turán densities and extremal numbers are known, and for most of them one also knows their (unique) extremal hypergraphs [FF89, DCF00, KM04, FS05, FPS05, KS05b, KS05a, Pik05, FPS06, Mub06, MP07, FMP08, Pik08]. Interestingly, the so-called stability method for graphs invented by Simonovits [Sim68] (and independently proved by Erdős [Erd68]), has been extensively used to establish many of the above exact results. The idea is to prove first some approximate result about an almost extremal F-free (hyper-)graph H and then to exploit imperfections in the structure to obtain exact results. The central tool for graphs is the following stability theorem:

Theorem 2.1 (Erdős [Erd68], Simonovits [Sim68])**.** *For every graph F of chromatic number $r \geq 3$ and for every $\varepsilon > 0$ there exist $\delta > 0$ and $n_0 \in \mathbb{N}$ such that if G is an F-free graph on $n \geq n_0$ vertices with $e(G) \geq \mathrm{ex}(n,F) - \delta n^2$ then there exists a partition of $V = V_1 \dot\cup \ldots \dot\cup V_{r-1}$ such that*

$$\sum_{i=1}^{r-1} e(V_i) < \varepsilon n^2.$$

Clearly, Theorem 2.1 asserts that graphs are 1-stable, where stability was introduced in Chapter 1, Definition 1.10. Recall, that a k-uniform hypergraph F is s-stable, if for every $\varepsilon > 0$ there exists an $\omega > 0$ and an integer n_0 such that for arbitrary F-free k-uniform hypergraphs $H_1, \ldots, H_{s+1}$ each of the same order $n \geq n_0$ and each having at least $\pi_F \binom{n}{k} - \omega n^k$ hyperedges, there are two which are ε-close. That is taking H_1 to be the Turán graph $T_{r-1}(n)$ and H_2 to be an "almost" extremal F-free graph, we uncover exactly the same statement about the existence of the desired vertex partition.

We denote by $\mathrm{Forb}(n,F)$ the family of all F-free labeled hypergraphs on n vertices. As a lower bound on its cardinality, we clearly have

$$|\mathrm{Forb}(n,F)| \geq 2^{\mathrm{ex}(n,F)},$$

as any subgraph of an extremal hypergraph for F is F-free. A graph property is a class of graphs that is closed under isomorphism. However when it comes to counting, we will distinguish between isomorphic graphs defined on the same vertex set, and say that they are labeled, which is the case for the family $\mathrm{Forb}(n, F)$.

2.1.5 Some further conventions and notations

Typically we will omit subscripts denoting the graphs under considerations and write $d(U, W)$ for $d_G(U, W)$. We also occasionally say subgraph meaning a *subhypergraph* and we sometimes use the word edges for *hyperedges*, to shorten our explanations. We will also tacitly assume that hypergraphs are k-uniform, and thus this letter will be used mostly for the uniformity throughout the thesis. For example, we will often write $E(U)$, $E(U, W)$, H instead of $E_H(U)$, $E_H(U, W)$, $H^{(k)}$ respectively. If we say that a (hyper-)graph G has n vertices, then we mostly identify its vertex set V with $[n]$. We also define

$$v_H := |V(H)| \quad \text{and} \quad e_H := |E(H)|$$

for a (hyper-)graph H.

From time to time we will ignore divisibility issues, as they do not affect our asymptotic considerations.

2.2 Szemerédi's regularity lemma

The regularity lemma [Sze78] roughly says that one can partition any graph into constantly many parts such that almost all of them look and behave as if they were random. For a survey on the regularity lemma for graphs and its further applications see [KS96] and [KSSS02].

The purpose of this section is twofold. Our first goal is to review the standard notation and state the regularity lemma for graphs, that we are going to use later in Chapter 4. Our second goal is to already outline the connections of the notion of an ε-regular pair to quasi-random graphs.

We say that a bipartite graph $G = (V_1 \dot\cup V_2, E)$, or simply (V_1, V_2), is ε-regular if all pairs of subsets $U_i \subseteq V_i$, with $|U_i| \geq \varepsilon |V_i|$, $i = 1, 2$, satisfy

$$|d_G(V_1, V_2) - d_G(U_1, U_2)| \leq \varepsilon.$$

An ε-regular pair (V_1, V_2) is called (ε, d)-regular if it has density at least d.

Now consider a partition $\{V_1, \dots, V_t\}$ of V such that $|V_1| \leq |V_2| \leq \dots \leq |V_t| \leq |V_1|+1$. We call such an equitable partition *ε-regular* if it satisfies the condition that all but $\varepsilon\binom{t}{2}$ pairs (V_i, V_j) are ε-regular, where $i \neq j, i, j \in [t]$. The vertex subsets V_i are referred to as *clusters* or *classes*.

The regularity lemma states then the following.

Theorem 2.2 (Regularity Lemma)**.** *For every integer $t_0 \geq 1$ and every $\varepsilon > 0$ there exist integers $T_0 = T_0(t_0, \varepsilon)$ and $n_0 = n_0(t_0, \varepsilon)$ such that every graph $G = (V, E)$ on at* *least n_0 vertices admits an ε-regular partition $V = V_1 \dot\cup \ldots \dot\cup V_t$ with $t_0 \le t \le T_0$.*

The bounds on $T(\varepsilon, t_0)$ obtained through the regularity lemma are tower exponential in ε^{-1} and this is necessary as shown by Gowers [Gow97].

In some versions of the regularity lemma, an exceptional set of size at most εn is allowed. However those vertices can be redistributed almost evenly among the vertex classes, so that most of the pairs are still ε-regular (with a slightly bigger ε).

A useful extension when dealing with graphs whose edges are colored with several colors is that one can apply a version of the regularity lemma and obtain an ε-regular partition simultaneously with respect to every color [KS96, Theorem 1.18].

Theorem 2.3 (Many-colors regularity lemma)**.** *For every $\varepsilon \in (0,1)$ and $r, t_0 \in \mathbb{N}$, there exist integers $n_0 = n_0(\varepsilon, r, t_0)$ and $T_0 = T_0(\varepsilon, r, t_0)$ such that the following holds. Every graph $G = (V, E)$ with $|V| \ge n_0$, whose edges are not necessarily properly r-colored: $E = E_1 \dot\cup \cdots \dot\cup E_r$, admits a partition of its vertex set: $V = V_1 \dot\cup \ldots \dot\cup V_t$, for some $t_0 \le t \le T_0$, which is ε-regular simultaneously with respect to every subgraph $G_i = (V, E_i)$, $i \in [r]$.*

Often, one first regularizes a graph G, and then defines a so-called cluster graph $R(\eta)$ for some $\eta > 0$ whose vertex set is $\{V_1, \ldots, V_t\}$ and whose edge set corresponds to (ε, η)-regular pairs. It turns out that the cluster graph inherits many properties of G and later in this chapter its notion will be generalized to cluster hypergraphs in different settings for hypergraph regularity lemmas.

Then for appropriate choices of $\varepsilon \ll \eta$ there is a statement called "counting lemma" which says that one can find roughly as many partite isomorphic copies in the subgraph of G that "underlies" $R(\eta)$ as one would find in a genuinely partite random graph with edge probabilities that equal the densities of the ε-regular pairs.

Theorem 2.4 (Counting lemma)**.** *For every γ and every graph F on ℓ vertices there exist $\varepsilon > 0$ and $m_0 \in \mathbb{N}$ such that the following holds. Let $V_1, \ldots, V_\ell$ be mutually disjoint sets such that a pair (V_i, V_j) is ε-regular whenever $ij \in E(F)$. Moreover, suppose that $|V_i| \ge m_0$ for all $i \in [\ell]$ and $d_{ij} = d(V_i, V_j)$. Then the number of partite isomorphic copies of F is within*

$$\prod_{ij \in E(F)} d_{ij} \prod_{i \in [\ell]} |V_i| \pm \gamma \prod_{i \in [\ell]} |V_i|$$

Now let us turn back to the random-looking pieces of an ε-regular partition. A useful fact about ε-regular pairs is that they "preserve" regularity.

Fact 2.5. *Let (V_1, V_2) be an ε-regular pair. If $U_i \subseteq V_i$ with $|U_i| \ge \alpha_i n_i$, where $\alpha_i \in (\varepsilon, 1)$, for $i \in [2]$, then (U_1, U_2) is $(\widetilde{\varepsilon}, d(V_1, V_2) - \varepsilon)$-regular with $\widetilde{\varepsilon} := \max\{2\varepsilon, \frac{\varepsilon}{\alpha_1}, \frac{\varepsilon}{\alpha_2}\}$.*

Another easily verifiable fact is that all but ε-proportion of the vertices of an ε-regular pair of density η have roughly the degree as one would expect in a random bipartite graph with edge probability η up to the error ε. Similar facts can be derived about common neighborhoods of the pairs of vertices and so on.

After having said these facts about ε-regular pairs, a connection to Theorem 1.1, mentioned in the introduction becomes apparent. In fact, an ε-regular pair resembles that what can be called a bipartite quasi-random graph. It is therefore useful to say a few words and even prove such a connection, as Chapter 3 will deal with its straightforward generalization to uniform hypergraphs.

In what follows, we give an argument of Gowers [Gow06]. So, suppose that we are provided with a bipartite graph $G = (X\dot{\cup}Y, E)$ with edge density d. Then, associate with G a function G from $X \times Y$ to $\{0,1\}$, where $G(x,y) = 1$ if $xy \in G$ and 0 otherwise. Further set $g\colon X \times Y \to [-1,1]$ with $g(x,y) := G(x,y) - d$. Then the following conditions are equivalent:

(i) For all $U \subset X, V \subset Y$ we have

$$\Big| e(U,V) - d|U||V| \Big| < \varepsilon_1 |X||Y|,$$

(ii) For all functions $a\colon X \to [0,1]$, $b\colon Y \to [-1,1]$ we have

$$\Big| \sum_{x\in X, y\in Y} g(x,y)a(x)b(y) \Big| < \varepsilon_2 |X||Y|, \text{and}$$

(iii)

$$\sum_{x,x'\in X, y,y'\in Y} g(x,y)g(x',y)g(x,y')g(x',y') < \varepsilon_3 |X|^2|Y|^2.$$

Here by equivalence, we mean that for every ε_i there exists an ε_j such that the corresponding implication holds. Typically, such a dependence is polynomial. Let us briefly prove these implications. Property (i) is clearly almost the same as the definition of ε-regularity. In fact, if G is ε_1-regular, then it satisfies property (i). This is a rather technical issue that we allow to take here all subsets. Indeed, taking subsets of small size does not say any meaningful things about the graph G. Thus, it can be easily seen that G is $\varepsilon_1^{1/3}$-regular in the sense discussed above.

$(i) \Leftrightarrow (ii)$ So, suppose that (ii) with $\varepsilon_2/4 = \varepsilon_1$ does not hold. Then, there exist functions a and b such that

$$\Big| \sum_{x\in X, y\in Y} g(x,y)a(x)b(y) \Big| \geq \varepsilon_2 |X||Y|.$$

We write $a = a^+ - a^-$, where $a^+(x) := \max\{a(x), 0\}$ and $a^-(x) := \max\{-a(x), 0\}$. Similarly for the function b: $b^+(y) := \max\{b(y), 0\}$ and $b^-(y) := \max\{-b(y), 0\}$. By the triangle inequality and without loss of generality we may assume:

$$\Big| \sum_{x\in X, y\in Y} g(x,y)a^+(x)b^+(y) \Big| \geq (\varepsilon_2/4)|X||Y|.$$

But the above sum is nothing else as the absolute value of the expectation of $e(U,V) - d|U||V|$, where U and V are random subsets of X, Y respectively, where the element $x \in X$ is chosen in U with probability $a^+(x)$ and similarly for V. Therefore, from the above inequality we deduce the existence of U and V with the same property which is a contradiction to (i). On the other hand, choosing a and b to be the indicator functions of U and V, we see that $(ii) \Rightarrow (i)$.

$(ii) \Rightarrow (iii)$ Suppose now that (ii) holds and we want to show (iii) with appropriate ε_3. For this we fix arbitrary $x' \in X, y' \in Y$ and we note that the sum

$$\sum_{x\in X, y\in Y} g(x,y)g(x',y)g(x,y')g(x',y')$$

is at most $\varepsilon_2|X||Y|$ via $a(x) := g(x,y')$ and $b(y) := g(x',y)g(x',y')$. Therefore, summing over all possible x' and y' we obtain the desired result with $\varepsilon_3 = \varepsilon_2$.

$(iii) \Rightarrow (ii)$ We will apply twice Cauchy-Schwarz inequality:

$$\begin{aligned}
\Big|\sum_{x\in X, y\in Y} g(x,y)a(x)b(y)\Big| \le \left(\sum_{x\in X} a(x)^2\right)^{1/2} \left(\sum_{x\in X}\left(\sum_{y\in Y} g(x,y)b(y)\right)^2\right)^{1/2} &\le \\
|X|^{1/2}\left(\sum_{y,y'\in Y, x\in X} g(x,y)g(x,y')b(y)b(y')\right)^{1/2} &\le \\
|X|^{1/2}\left(\sum_{y,y'\in Y}(b(y)b(y'))^2\right)^{1/4}\left(\sum_{y,y'\in Y}\left(\sum_{x\in X} g(x,y)g(x,y')\right)^2\right)^{1/4} &\le \\
(|X||Y|)^{1/2}\left(\sum_{x,x'\in X, y,y'\in Y} g(x,y)g(x',y)g(x,y')g(x',y')\right)^{1/4} &\le \varepsilon_3^{1/4}|X||Y|,
\end{aligned}$$

thus, setting $\varepsilon_2 = \varepsilon_3^{1/4}$ finishes the proof.

One might also notice that while the property (iii) is checkable in polynomial time, the property (i) is co-NP-complete [ADL$^+$94]. Making the use of the approximate equivalences above it is possible to give a constructive proof of regularity lemma [ADL$^+$94]. A straightforward approach would be to "derandomize" above property (iii) to find an obstruction to property (i).

2.3 The weak hypergraph regularity lemma

A straightforward generalization of the notion of an ε-regular pair to that of an ε-regular k-tuple for a k-uniform hypergraph is the following. We say the k-tuple $(V_1,\dots,V_k)$ of pairwise disjoint subsets $V_1,\dots,V_k \subseteq V(H)$ of a k-uniform hypergraph H is *ε-regular* if

$$|d(U_1,\dots,U_k) - d(V_1,\dots,V_k)| \le \varepsilon$$

for all k-tuples of subsets $U_1 \subset V_1, \dots, U_k \subset V_k$ satisfying $|U_1| \geq \varepsilon |U_1|, \dots |U_k| \geq \varepsilon |V_k|$.

A k-tuple $(V_1, \dots, V_k)$ of mutually disjoint subsets $V_1, \dots, V_k \subseteq V(H)$, which is not ε-regular, is called *ε-irregular.*

Though the notion of weak regularity is not sufficient to imply a general counting lemma it was shown in [KNRS10] that it is strong enough to imply a counting lemma for linear hypergraphs:

Lemma 2.6 (Counting lemma for linear hypergraphs [KNRS10])**.** *For all integers $\ell \geq k \geq 2$ and every γ, there exist $\varepsilon = \varepsilon(\ell, k, \gamma) > 0$ and $m_0 = m_0(\ell, k, \gamma)$ so that the following holds.*

Let $F = ([\ell], E(F))$ be a linear k-uniform hypergraph and let $H = (V_1 \dot\cup \dots \dot\cup V_\ell, E)$ be an ℓ-partite, k-uniform hypergraph where $|V_1|, \dots, |V_\ell| \geq m_0$. Suppose, moreover, that for all edges $f \in E(F)$, the k-tuple $(V_i)_{i \in f}$ is ε-regular and has density d_f. Then the following holds:

$$N_F(V_1, \dots, V_\ell) = \prod_{f \in E(F)} d_f \prod_{i \in [\ell]} |V_i| \pm \gamma \prod_{i \in [\ell]} |V_i| \, .$$

□

An *ε-regular partition* of a vertex set $V(H)$ has the following properties:

(i) $V(H) = V_1 \dot\cup \cdots \dot\cup V_t$, and

(ii) $||V_i| - |V_j|| \leq 1$ for all i, j, and

(iii) for all but at most $\varepsilon \cdot \binom{t}{k}$ many k-element subsets $\{i_1, \dots, i_k\} \subseteq [t]$ the k-tuple $(V_{i_1}, \dots, V_{i_k})$ is ε-regular.

Weak hypergraph regularity lemma, which is a straightforward extension of Szemerédi's regularity lemma for graphs [Sze78], states then the following.

Theorem 2.7 (Weak hypergraph regularity lemma)**.** *For all k, $t_0 \in \mathbb{N}$ and all $\varepsilon > 0$ there is a $T_0 = T_0(t_0, \varepsilon)$ and an n_0 such that for all $n \geq n_0$, any k-uniform hypergraph H on n vertices admits an ε-regular partition with the number of classes t satisfying $t_0 \leq t \leq T_0$.* □

Its proof follows the lines of the original proof of Szemerédi (see, e.g., [Chu91, FR92, Ste90]). We will also make use of the colored version of it:

Theorem 2.8. *Let $k \geq 2$, $r \geq 1$ and $t_0 \geq 1$ be fixed integers. For every $\varepsilon > 0$, there exist $T_0 = T_0(r, t_0, \varepsilon)$ and $n_0 = n_0(r, t_0, \varepsilon)$ such that for every k-uniform hypergraph H on $n \geq n_0$ vertices, whose hyperedges are r-colored, i.e., $E(H) = E_1 \dot\cup \cdots \dot\cup E_r$, there exists a partition $V(H) = V_1 \dot\cup \cdots \dot\cup V_t$, with $t_0 \leq t \leq T_0$, which is ε-regular simultaneously with respect to each subhypergraph $H_i = (V, E_i)$, $i \in [r]$.* □

For a hypergraph H and a regular partition of its vertex set we, similarly to the graph case, use the concept of a *cluster hypergraph.*

Definition 2.9. *For a hypergraph H, an ε-regular partition $V(H) = V_1\dot{\cup}\cdots\dot{\cup}V_t$ of its vertex set, and a number $\gamma > 0$ let $H(\gamma)$ be the cluster hypergraph with vertex set $V(H(\gamma)) = [t]$ and the set $E(H(\gamma))$ of hyperedges, where for $1 \leq i_1 < \cdots < i_k \leq t$ it is $\{i_1, \ldots, i_k\} \in E(H(\gamma))$ if and only if the k-tuple $(V_{i_1}, \ldots, V_{i_k})$ is ε-regular and its density satisfies $d_H(V_{i_1}, \ldots, V_{i_k}) \geq \gamma$.*

Thus, Lemma 2.6 in disguise implies the following

Lemma 2.10. *Let F be a fixed linear k-uniform hypergraph. For each $\eta > 0$ there exists $\varepsilon = \varepsilon(\eta) > 0$ and an integer $m_0 = m_0(\eta)$ such that for every positive integer t the following holds.*

Let H be a k-uniform hypergraph with an ε-regular partition $V(H) = V_1\dot{\cup}\cdots\dot{\cup}V_t$, where $|V_i| \geq m_0$ for every $i \in [t]$. If the cluster hypergraph $H(\eta)$ contains a subhypergraph F, then the hypergraph H contains a subhypergraph F too. □

2.4 The strong hypergraph regularity lemma of Rödl and Schacht

Before we state the regularity and the counting lemmas [RS07b, RS07a], we introduce some notation.

2.4.1 Complexes

Given integers $j \leq \ell$ and mutually disjoint vertex sets $V_1, \ldots, V_\ell$ each of size m, then an (m, ℓ, j)*-hypergraph* $H^{(j)}$ on $V_1 \cup \cdots \cup V_\ell$ is any subhypergraph of $K_\ell^{(j)}(V_1, \ldots, V_\ell)$. Note that the vertex partition $V_1 \cup \cdots \cup V_\ell$ is an $(m, \ell, 1)$-hypergraph $H^{(1)}$. For $j \leq i \leq \ell$ and a set $\Lambda_i \in [\ell]^i$, we denote by $H^{(j)}[\Lambda_i] = H^{(j)}[\bigcup_{\lambda\in\Lambda_i} V_\lambda]$ the subhypergraph of the (m, ℓ, j)-hypergraph $H^{(j)}$ induced on $\bigcup_{\lambda\in\Lambda_i} V_\lambda$.

For an (m, ℓ, j)-hypergraph $H^{(j)}$ and an integer i, $j \leq i \leq \ell$, we denote by $\mathcal{K}_i(H^{(j)})$ the set of all i-subsets of $V(H^{(j)})$ which span complete subhypergraphs in $H^{(j)}$ on i vertices. Note that $|\mathcal{K}_i(H^{(j)})|$ is the number of all copies of $K_i^{(j)}$ in $H^{(j)}$.

Given an $(m, \ell, j-1)$-hypergraph $H^{(j-1)}$ and an (m, ℓ, j)-hypergraph $H^{(j)}$ such that $V(H^{(j)}) \subseteq V(H^{(j-1)})$, we say that a hyperedge J of $H^{(j)}$ *belongs to* $H^{(j-1)}$ if $J \in \mathcal{K}_j(H^{(j-1)})$, i.e., J corresponds to a clique of order j in $H^{(j-1)}$. Moreover, $H^{(j-1)}$ *underlies* $H^{(j)}$ if $H^{(j)} \subseteq \mathcal{K}_j(H^{(j-1)})$, i.e., every hyperedge of $H^{(j)}$ belongs to $H^{(j-1)}$. This brings us to the notion of a *complex*. Let $m \geq 1$ and $\ell \geq h \geq 1$ be integers. An (m, ℓ, h)*-complex* $\boldsymbol{\mathcal{H}}$ is a collection of (m, ℓ, j)-hypergraphs $\{H^{(j)}\}_{j=1}^h$ such that

(*a*) $H^{(1)}$ is an $(m, \ell, 1)$-hypergraph, i.e., $V(H^{(1)}) = V_1 \cup \cdots \cup V_\ell$ with $|V_i| = m$ for $i \in [\ell]$, and

(*b*) $H^{(j-1)}$ underlies $H^{(j)}$ for $2 \leq j \leq h$, i.e., $H^{(j)} \subseteq \mathcal{K}_j(H^{(j-1)})$.

Now we proceed with the notion of relative density of a j-uniform hypergraph with respect to a $(j-1)$-uniform hypergraph on the same vertex set. For a given j-uniform hypergraph $H^{(j)}$ and a $(j-1)$-uniform hypergraph $H^{(j-1)}$ on the same vertex set, we define the *density of $H^{(j)}$ with respect to $H^{(j-1)}$* as

$$d(H^{(j)}|H^{(j-1)}) = \begin{cases} \frac{|H^{(j)}\cap\mathcal{K}_j(H^{(j-1)})|}{|\mathcal{K}_j(H^{(j-1)})|} & \text{if} \quad \left|\mathcal{K}_j(H^{(j-1)})\right| > 0 \\ 0 & \text{otherwise}. \end{cases}$$

We also use a notion of regularity of an (m,j,j)-hypergraph with respect to an $(m,j,j-1)$-hypergraph. Let reals $\varepsilon > 0$ and $d_j \geq 0$ be given along with an (m,j,j)-hypergraph $H^{(j)}$ and an underlying $(m,j,j-1)$-hypergraph $H^{(j-1)}$. We say $H^{(j)}$ is (ε, d_j)*-regular* with respect to $H^{(j-1)}$ if whenever $Q^{(j-1)} \subseteq H^{(j-1)}$ satisfies

$$|\mathcal{K}_j(Q^{(j-1)})| \geq \varepsilon|\mathcal{K}_j(H^{(j-1)})|, \quad \text{then} \quad d(H^{(j)}|Q^{(j-1)}) = d_j \pm \varepsilon.$$

More generally, we extend the notion of (ε, d_j)-regularity from (m,j,j)-hypergraphs to (m,ℓ,j)-hypergraphs $H^{(j)}$. We say that an (m,ℓ,j)-hypergraph $H^{(j)}$ is (ε, d_j)*-regular* with respect to an $(m,\ell,j-1)$-hypergraph $H^{(j-1)}$ if for every j-subset $\Lambda_j \in [\ell]^j$ the restriction $H^{(j)}[\Lambda_j] = H^{(j)}\big[\bigcup_{\lambda\in\Lambda_j} V_\lambda\big]$ is (ε, d_j)-regular with respect to to the restriction $H^{(j-1)}[\Lambda_j] = H^{(j-1)}\big[\bigcup_{\lambda\in\Lambda_j} V_\lambda\big]$.

We sometimes write ε*-regular* to mean $\big(\varepsilon, d(H^{(j)}|H^{(j-1)})\big)$-regular and we also omit the m in (m,ℓ,h)-complex and (m,ℓ,h)-hypergraph.

Finally, a regular complex is defined as follows.

Definition 2.11 (**$(\varepsilon, \boldsymbol{d})$-regular complex**). *Let $\varepsilon > 0$ and let $\boldsymbol{d} = (d_2, \dots, d_h)$ be a vector of non-negative reals. An (m,ℓ,h)-complex $\boldsymbol{\mathcal{H}} = \{H^{(j)}\}_{j=1}^h$ is* $(\varepsilon, \boldsymbol{d})$-regular *if $H^{(j)}$ is (ε, d_j)-regular with respect to $H^{(j-1)}$ for $j = 2, \dots, h$.*

2.4.2 Partitions

The regularity lemmas [RS07b, RS07a] provide a well-structured family of partitions $\mathscr{P} = \{\mathscr{P}^{(1)}, \dots, \mathscr{P}^{(k-1)}\}$, where $\mathscr{P}^{(i)}$ is a partition of the set of all i-subsets of some vertex set, which we define below inductively.

Let k be a fixed integer and let $\mathscr{P}^{(1)} = \{V_1, \dots, V_{|\mathscr{P}^{(1)}|}\}$ be a partition of some vertex set V. For every j, $1 \leq j \leq k$, let $\mathrm{C}ross_j(\mathscr{P}^{(1)})$ be the family of all crossing j-subsets J of V, i.e., the set of all j-subsets which satisfy $|J \cap V_i| \leq 1$ for every $V_i \in \mathscr{P}^{(1)}$.

Suppose that partitions $\mathscr{P}^{(i)}$ of $\mathrm{C}ross_i(\mathscr{P}^{(1)})$ into sets of (i,i)-hypergraphs, i.e., i-partite i-uniform hypergraphs, have been defined for $i = 1, \dots, j-1$. Then for every $(j-1)$-subset I in $\mathrm{C}ross_{j-1}(\mathscr{P}^{(1)})$ there exists a unique $\mathcal{P}^{(j-1)} = \mathcal{P}^{(j-1)}(I) \in \mathscr{P}^{(j-1)}$ so that $I \in \mathcal{P}^{(j-1)}$. Moreover, for every j-subset J in $\mathrm{C}ross_j(\mathscr{P}^{(1)})$ we define the *polyad of J* to be

$$\hat{\mathcal{P}}^{(j-1)}(J) = \bigcup \left\{\mathcal{P}^{(j-1)}(I)\colon\ I \in [J]^{j-1}\right\}.$$

In other words, $\hat{\mathcal{P}}^{(j-1)}(J)$ is the unique collection of j partition classes of $\mathscr{P}^{(j-1)}$ in which J spans a copy of $K_j^{(j-1)}$. Observe that $\hat{\mathcal{P}}^{(j-1)}(J)$ can be viewed as a $(j,j-1)$-hypergraph, i.e., a j-partite, $(j-1)$-uniform hypergraph. More generally, for $1 \leq i < j$, we set

$$\hat{\mathcal{P}}^{(i)}(J) = \bigcup \left\{ \mathcal{P}^{(i)}(I) \colon I \in [J]^i \right\} \qquad \text{and} \qquad \boldsymbol{\mathcal{P}}(J) = \{\hat{\mathcal{P}}^{(i)}(J)\}_{i=1}^{j-1}. \tag{2.3}$$

We also refer to $\boldsymbol{\mathcal{P}}(J)$ as the polyad of J and it will always be clear from the context which definition is meant.

Next, we define $\hat{\mathscr{P}}^{(j-1)}$, the family of all polyads by

$$\hat{\mathscr{P}}^{(j-1)} = \{\hat{\mathcal{P}}^{(j-1)}(J) \colon J \in \mathrm{C}ross_j(\mathscr{P}^{(1)})\}.$$

Note that two polyads $\hat{\mathcal{P}}^{(j-1)}(J)$ and $\hat{\mathcal{P}}^{(j-1)}(J')$ are not necessarily distinct for different j-subsets J and J'. We view $\hat{\mathscr{P}}^{(j-1)}$ as a set and, consequently,

$$\{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)}) \colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$$

is a partition of $\mathrm{C}ross_j(\mathscr{P}^{(1)})$. The structural requirement on the partition $\mathscr{P}^{(j)}$ of $\mathrm{C}ross_j(\mathscr{P}^{(1)})$ that one makes is that the set of cliques spanned by a polyad in $\hat{\mathscr{P}}^{(j-1)}$ is sub-partitioned in $\mathscr{P}^{(j)}$ and every partition class in $\mathscr{P}^{(j)}$ belongs to precisely one polyad in $\hat{\mathscr{P}}^{(j-1)}$, and we say that $\mathscr{P}^{(j)}$ *refines* $\{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)}) \colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\}$. Note by our inductive definition that $\boldsymbol{\mathcal{P}}(J)$ defined in (2.3) is a $(j, j-1)$-complex.

Definition 2.12 (family of partitions)**.** *Suppose V is a set of vertices, $k \geq 2$ is an integer and $\boldsymbol{a} = (a_1, \ldots, a_{k-1})$ is a vector of positive integers. We say $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}) = \{\mathscr{P}^{(1)}, \ldots, \mathscr{P}^{(k-1)}\}$ is a family of partitions on V, if it satisfies the following:*

(*i*) *$\mathscr{P}^{(1)}$ is a partition of V into a_1 classes, and*

(*ii*) *for $j = 2, \ldots, k-1$, $\mathscr{P}^{(j)}$ is a partition of $Cross_j(\mathscr{P}^{(1)})$ satisfying:*

$$\begin{aligned} &\mathscr{P}^{(j)} \quad \textit{refines} \quad \{\mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)}) \colon \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}\} \\ \textit{and} \quad &\left|\{\mathcal{P}^{(j)} \in \mathscr{P}^{(j)} \colon \mathcal{P}^{(j)} \subseteq \mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})\}\right| = a_j \quad \textit{for every} \quad \hat{\mathcal{P}}^{(j-1)} \in \hat{\mathscr{P}}^{(j-1)}. \end{aligned}$$

Moreover, we say $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ is t-bounded, if $\max\{a_1, \ldots, a_{k-1}\} \leq t$.

2.4.3 Equitability and regular hypergraphs

In this subsection we introduce the notion of equitability.

Definition 2.13 ($(\eta, \varepsilon, \boldsymbol{a})$-equitable)**.** *Suppose V is a set of n vertices, η and ε are positive reals, $\boldsymbol{a} = (a_1, \ldots, a_{k-1})$ is a vector of positive integers, and a_1 divides n.*

*We say a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}) = \{\mathscr{P}^{(1)}, \ldots, \mathscr{P}^{(k-1)}\}$ on V is $(\eta, \varepsilon, \boldsymbol{a})$-*equitable *if it satisfies the following:*

(*a*) *$\left|[V]^k \setminus Cross_k(\mathscr{P}^{(1)})\right| \leq \eta\binom{n}{k}$, and*

(*b*) *$\mathscr{P}^{(1)} = \{V_i \colon i \in [a_1]\}$ is an equitable vertex partition, i.e., $|V_i| = |V|/a_1$ for each $i \in [a_1]$, and* (*c*) *for every k-subset $K \in Cross_k(\mathscr{P}^{(1)})$ we have that $\boldsymbol{\mathcal{P}}(K) = \{\hat{\mathcal{P}}^{(j)}(K)\}_{j=1}^{k-1}$ is an $(\varepsilon, \boldsymbol{d})$-regular $(n/a_1, k, k-1)$-complex, where $\boldsymbol{d} = (1/a_2, \dots, 1/a_{k-1})$.*

Note that from equitability one obtains an implicit bound $a_1 > 1/(2\eta)$ for n sufficiently large.

Now we extend the definition of (ε, d_j)-regularity.

Definition 2.14 ((δ_k, d_k, f)-regular hypergraph)**.** *Let δ_k and d_k be positive reals and f be a positive integer. Suppose $H^{(k-1)}$ is an $(m, k, k-1)$-hypergraph spanning at least one $K_k^{(k-1)}$. We say an (m, k, k)-hypergraph $H^{(k)}$ is* (δ_k, d_k, f)-regular *with respect to $H^{(k-1)}$ if for every collection $\boldsymbol{\mathcal{Q}}^{(k-1)} = \{Q_1^{(k-1)}, \dots, Q_f^{(k-1)}\}$ of not necessarily disjoint subhypergraphs of $H^{(k-1)}$ which satisfy*

$$\Big|\bigcup_{i\in[f]} \mathcal{K}_k(Q_i^{(k-1)})\Big| \geq \delta_k \Big|\mathcal{K}_k(H^{(k-1)})\Big| > 0\,,$$

we have

$$\frac{|H^{(k)} \cap \bigcup_{i\in[f]} \mathcal{K}_k(Q_i^{(k-1)})|}{|\bigcup_{i\in[f]} \mathcal{K}_k(Q_i^{(k-1)})|} = d_k \pm \delta_k\,.$$

We write $(\delta_k, *, f)$-regular *to mean $\big(\delta_k, d(H^{(k)}|H^{(k-1)}), f\big)$-regular. Moreover, if $f = 1$, then a $(\delta_k, d_k, 1)$-regular hypergraph is (ε, d_k)-regular with $\varepsilon = \delta_k$.*

Next we say when a hypergraph is regular with respect to a given family of partitions.

Definition 2.15 ($(\delta_k, *, f)$-regular with respect to $\mathscr{P}$)**.** *Let δ_k be a positive real and f a positive integer. Let $H^{(k)}$ be a k-uniform hypergraph with vertex set V and $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$ be a family of partitions on V. We say $H^{(k)}$* is $(\delta_k, *, f)$-regular with respect to $\mathscr{P}$*, if*

$$\Big|\bigcup\Big\{\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})\colon\ \hat{\mathcal{P}}^{(k-1)} \in \hat{\mathscr{P}}^{(k-1)} \text{ and } H^{(k)} \text{ is not } (\delta_k, *, f)\text{-regular with respect to } \hat{\mathcal{P}}^{(k-1)}\Big\}\Big| \leq \delta_k |V|^k\,.$$

If $H^{(k)}$ is $(\delta_k, *, f)$-regular with respect to $\hat{\mathcal{P}}^{(k-1)}$ then we call the polyad $\hat{\mathcal{P}}^{(k-1)}$ and also $\boldsymbol{\mathcal{P}}(J)$ regular, where $J \in \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$. Note, that $\boldsymbol{\mathcal{P}}(J) = \boldsymbol{\mathcal{P}}(J')$ for all J and J' in $\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)})$.

2.4.4 The regularity and counting lemmas

Finally, we state the regularity lemma [RS07b] we are going to use (see for example Lemma 23 in [RS07b]).

Theorem 2.16 (Regularity lemma)**.** *Let $k \geq 2$ and $c \geq 1$ be fixed integers. For all positive constants η and δ_k, and all functions $f\colon \mathbb{N}^{k-1} \to \mathbb{N}$ and $\delta\colon \mathbb{N}^{k-1} \to (0, 1]$ there are integers t_0 and n_0 so that the following holds.*

For every k-uniform hypergraph H, which is a hyperedge-disjoint union[1] *of c k-uniform hypergraphs $H = H_1 \dot\cup \ldots \dot\cup H_c$ with $|V(H)| = n \geq n_0$ such that $t_0!$ divides n, there exists a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ so that*

(i) $\mathscr{P}$ is $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable and t_0-bounded and

(ii) H_i is $(\delta_k, *, f(\boldsymbol{a}^{\mathscr{P}}))$-regular with respect to $\mathscr{P}$ for every color $i \in [c]$.

We use the following lemma, its proof can be derived from Theorem 1.3 in [RS07a], we also refer the interested reader to Chapter 9 of [Sch04].

Theorem 2.17 (Counting lemma)**.** *For any integer $k \geq 2$, every k-uniform hypergraph F and every positive constant $d_k > 0$, there exists $\delta_k > 0$ such that for every $d_{k-1}, \ldots, d_2 > 0$ with $1/d_i \in \mathbb{N}$ for every $i = 2, \ldots, k-1$ there are constants $\delta = \delta(d_2, \ldots, d_{k-1}) > 0$ and positive integers $f = f(d_2, \ldots, d_{k-1})$ and m_0 such that the following holds. Let H be a k-uniform hypergraph on $n \geq a_1 m_0$ vertices and let $\mathscr{P}(k-1, (a_1, 1/d_2, \ldots, 1/d_{k-1})) = \{\mathscr{P}^{(1)}, \ldots, \mathscr{P}^{(k-1)}\}$ be a family of partitions.*

If for a copy F' of F in the complete k-uniform hypergraph with vertex set as $V(H)$ the following conditions are satisfied

(i) For every $e \in E(F')$ the polyad $\boldsymbol{\mathcal{P}}_e = \{\hat{\mathcal{P}}_e^{(i)}\}_{i=1}^{k-1}$ with $e \in \mathcal{K}_k(\hat{\mathcal{P}}_e^{(k-1)})$ is a $(\delta, (d_2, \ldots, d_{k-1}))$-regular $(n/a_1, k, k-1)$-complex, and

(ii) for every $e \in E(F')$ the hypergraph H is (δ_k, d_e, f)-regular with respect to $\hat{\mathcal{P}}_e^{(k-1)}$ for some $d_e \geq d_k$,

then H contains at least one copy of F.

Roughly speaking, this theorem says that if a collection of sufficiently regular complexes of H is in a natural correspondence to the hyperedges of some fixed hypergraph F, then the given hypergraph H must contain a copy of F.

2.4.5 Cluster hypergraphs and slices

Similarly as in the graph case, we study the "cluster hypergraph" of a given family of partitions. However, in the hypergraph setting the natural cluster hypergraph is a "multihypergraph" and for our purposes it suffices to analyze an appropriate subhypergraph without multiple hyperedges. Such a representative we call a *slice*.

In the following we describe a slice more formally. For a given family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a})$, for every pair (V_i, V_j) of two vertex classes from $\mathscr{P}^{(1)}$ we choose precisely one element $\mathcal{P}$ from $\mathscr{P}^{(2)}$ such that $\mathcal{P} \subseteq \mathcal{K}_2(V_i, V_j)$. More generally, for every polyad $\hat{\mathcal{P}}^{(j-1)}$ (which is a $(j, j-1)$-hypergraph) formed by elements from the slice we select precisely one (j,j)-hypergraph $\mathcal{P}^{(j)}$ with $\mathcal{P}^{(j)} \subseteq \mathcal{K}_j(\hat{\mathcal{P}}^{(j-1)})$ from $\mathscr{P}^{(j)}$ for the slice. Note that there are exactly

$$\prod_{i=2}^{k-1} a_i^{\binom{a_1}{i}} \tag{2.4}$$

[1] equivalently, one can think of hyperedges of H being colored with c colors

slices in the family $\mathscr{P}(k-1,\boldsymbol{a})$.

Define for every slice $\mathcal{S}$ a $|\mathscr{P}^{(1)}|$-partite k-uniform hypergraph $H(\mathcal{S})$ with the vertex partition $\mathscr{P}^{(1)}$ whose hyperedges are exactly those k-sets that correspond to regular polyads $\boldsymbol{\mathcal{P}}(J)$, contained in the slice $\mathcal{S}$ and such that $H \cap \mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}(J))$ has sufficiently large density (with respect to $\mathcal{K}_k(\hat{\mathcal{P}}^{(k-1)}(J))$), say, at least d_k. Then a special case of Theorem 2.17 above states that if $H(\mathcal{S})$ contains some copy of F, then $H^{(k)}$ must contain a copy of F too.

Remark 2.18. Gowers proved in [Gow06, Gow07] regularity and counting lemmas of similar strength. He calls complexes k-partite chains and, more generally, the family of partitions is referred to as chain decomposition. However, his definition of regularity is defined via relative "octahedral quasi-randomness" and ε-regular chains are called ε-quasi-random. These definitions are nevertheless equivalent as shown for 3-uniform hypergraphs in [NPRS09].

2.5 Tools from probability theory

We will be using tools from discrete probability theory to bound various events involving discrete random variables. In the following we assume that a finite probability space $(\Omega, \mathbb{P})$ is given (such space will always be either clear from the context or specified explicitly).

For us the probability space $\mathcal{G}(n,p)$ will be important. There, every edge of K_n is included in the random graph independently of the other edges with probability p.

In the following we briefly review the union bound and the first, the second and the exponential moment methods and Janson's inequality as well. The proofs of results stated below in this section can be found in any standard book on probabilistic methods or about random graphs, see for example [AS08, Bol01, JŁR00].

Let $E_1, E_2, \dots$ be some events. Then the following inequality holds

$$\mathbb{P}(\bigcup_i E_i) \leq \sum_i \mathbb{P}(E_i), \tag{2.5}$$

and we refer to it as a union bound.

For a nonnegative random variable X and for any positive real number a we always have

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}, \tag{2.6}$$

and this fact is called Markov's inequality (first moment method).

Let X be a random variable and let $\lambda > 0$, then Chebyshev's inequality asserts

$$\mathbb{P}(|X - \mathbb{E}(X)| \geq \lambda \mathrm{Var}(X)^{1/2}) \leq \frac{1}{\lambda^2}. \tag{2.7}$$

This is also known as the second moment method.

We refer to the following theorem as Chernoff's inequality [Che52]:

Theorem 2.19. *Let X_i be jointly independent $\{0,1\}$-random variables with $\mathbb{P}(X_i = 1) = p_i, i = 1, \dots, n$, and let $X = \sum_{i=1}^n X_i$. Then for every $t \geq 0$ the following holds:*

$$\mathbb{P}(X \geq \mathbb{E}(X)+t) \leq \exp\left(-\frac{t^2}{2(\mathbb{E}(X) + t/3)}\right) \quad \textit{and} \quad \mathbb{P}(X \leq \mathbb{E}(X)-t) \leq \exp\left(-\frac{t^2}{2\mathbb{E}(X)}\right)$$

Finally, to deal with random variables which do not enjoy the independence property, and still to be able to give good bounds for the lower tail probability $\mathbb{P}(X \leq \mathbb{E}(X) - t)$, one might use Janson's inequality [Jan90].

Theorem 2.20. *Let $t_1, \dots t_n$ be jointly independent indicator random variables, let $\mathcal{A} \subset 2^{[n]} \setminus \{\varnothing\}$, and set $X := \sum_{A \in \mathcal{A}} \prod_{i \in A} t_i$. Then the following holds for every $t \geq 0$:*

$$\mathbb{P}(X \leq \mathbb{E}(X) - t) \leq \exp\left(-\frac{t^2}{2\Delta}\right),$$

where

$$\Delta := \sum_{A,B \in \mathcal{A}\colon\, A \cap B \neq \varnothing} \mathbb{E}\Big(\prod_{i \in A \cup B} t_i\Big).$$

Note, that setting $t = \mathbb{E}(X)$ in the above theorem, one obtains

$$\mathbb{P}(X = 0) \leq \exp\left(-\frac{\mathbb{E}(X)^2}{2\Delta}\right).$$

3 Weak quasi-randomness for uniform hypergraphs

We establish a generalization of theorem of Chung, Graham and Wilson on quasi-random graphs [CGW89], Theorem 1.1, for k-uniform hypergraphs based on DISC_d, which is Theorem 1.3, discussed in the next section. In Section 3.1.2 we present a short proof of Corollary 1.4, which is a consequence of this generalization for graphs. We will also verify the equivalence of another property for k-uniform hypergraphs, which is inspired by Theorem 1.2 and which we discuss in Section 3.1.3 (see Theorem 3.3) and then prove in Section 3.3. Then we show the equivalence of several partite variants of DISC_d (see Theorem 3.4 in Section 3.1.4 and the proofs in Section 3.4). Finally we present a simple strong refutation algorithm, Theorem 1.6, see Section 3.5.

3.1 Equivalent properties for weak quasi-randomness

The theorem of Chung, Graham and Wilson [CGW89] shows equivalence of some seven properties for sequences of graphs. In the section below we discuss extensions of properties P_1, P_2, P_6 and P_7 to k-uniform hypergraphs, which all turn out to be equivalent to the following straightforward generalization of the property P_4:

$\mathrm{DISC}_d(\delta)$ We say a k-uniform hypergraph H_n on n vertices has $\mathrm{DISC}_d(\delta)$ for d, $\delta > 0$, if

$$e(U) = d\binom{|U|}{k} \pm \delta n^k \quad \text{for all} \quad U \subseteq V(H_n)\,.$$

3.1.1 Generalization of Theorem 1.1

We will suggest extensions of properties P_1, P_2, P_6, and P_7 to k-uniform hypergraphs which all turn out to be equivalent to DISC_d (the analogue of P_4 in this context).

Extension of P_1

We start with property P_1. This property asserts that the number of induced copies of a fixed graph F in G_n is asymptotically the same as in the random graph $G(n, 1/2)$. It is well known that DISC_d does not imply such a property for $k \geq 3$ as the following example shows: let H_n be the 3-graph whose edges are formed by the triangles of the random graph $G(n, 1/2)$. Chernoff type estimates, Theorem 2.19, show that H_n satisfies $\mathrm{DISC}_{1/8}$ with high probability. On the other hand, the number of labeled (not necessarily induced) copies of $K^{(3)}_{1,1,2}$ (the 3-graph with two edges on four vertices) in H_n is $\sim n^4/32$, which is twice as much as the "right" number $(1/8)^2 n^4$. Moreover, the number of labeled, induced copies of $K^{(3)}_{1,1,2}$ in H_n is $\sim n^4/64$, while the "right" number would be $49n^4/64^2$.

However, it was shown in [KNRS10], that k-uniform hypergraphs having $\mathrm{DISC}_d(\delta)$ for sufficiently small δ must contain approximately the same number of copies of any fixed linear k-uniform hypergraph F as a genuine random k-uniform hypergraph of the same density. Recall that a *linear* k-uniform hypergraph F was defined as having no pair of edges which intersect in two or more vertices. In other words, the property DISC_d implies the following counting-lemma-type property,

$\mathrm{CL}_d(F,\varepsilon)$ We say a k-uniform hypergraph H_n on n vertices has $\mathrm{CL}_d(F,\varepsilon)$ for a given linear k-uniform hypergraph F on ℓ vertices and d, $\varepsilon > 0$, if

$$N_F(H_n) = d^{e(F)} n^\ell \pm \varepsilon n^\ell\,,$$

where $N_F(H_n)$ denotes the number of labeled copies of F in H_n.

For a property $P_{x_1,\dots,x_p}(\alpha_1,\dots,\alpha_r)$ of k-uniform hypergraphs we say a sequence $(H_n)_{n\in\mathbb{N}}$ of k-uniform hypergraphs with $|V(H_n)| = n$ *has* or *satisfies* $P_{x_1,\dots,x_p}$, if for all choices of the parameters $\alpha_1,\dots,\alpha_r$ there exists an n_0 such that H_n satisfies $P_{x_1,\dots,x_p}(\alpha_1,\dots,\alpha_r)$ for all $n \geq n_0$. Note that the parameters $x_1,\dots,x_p$ are fixed for this definition and the fixed parameters always appear as subscripts on the name of the property. Moreover, the parameters $x_1,\dots,x_p$ and $\alpha_1,\dots,\alpha_r$ might be of different types, like k-uniform hypergraphs, integers, or real numbers. For example, in CL_d the parameter α_1 is an arbitrary linear k-uniform hypergraph, while x_1 and α_2 are positive reals. Furthermore, for two properties $P_{x_1,\dots,x_p}(\alpha_1,\dots,\alpha_r)$ and $Q_{y_1,\dots,y_q}(\beta_1,\dots,\beta_s)$ we say $P_{x_1,\dots,x_p}$ *implies* $Q_{y_1,\dots,y_q}$ ($P_{x_1,\dots,x_p} \Rightarrow Q_{y_1,\dots,y_q}$), if every sequence of k-uniform hypergraphs $(H_n)_{n\in\mathbb{N}}$ that satisfies property $P_{x_1,\dots,x_p}$ also satisfies property $Q_{y_1,\dots,y_q}$. Moreover, properties $P_{x_1,\dots,x_p}$ and $Q_{y_1,\dots,y_q}$ are called equivalent if $P_{x_1,\dots,x_p} \Rightarrow Q_{y_1,\dots,y_q}$ and $Q_{y_1,\dots,y_q} \Rightarrow P_{x_1,\dots,x_p}$. With this notation, the aforementioned result from [KNRS10] states that

$$\mathrm{DISC}_d \text{ implies } \mathrm{CL}_d. \tag{3.1}$$

The discussion above suggests that the "right" extension of P_1 in our context involves linear k-uniform hypergraphs, which leads to the following definition for the induced-counting-lemma-type property.

$\mathrm{ICL}_d(F',F,\varepsilon)$ We say a k-uniform hypergraph H_n on n vertices has $\mathrm{ICL}_d(F',F,\varepsilon)$ for given linear k-uniform hypergraphs $F' \subseteq F$ with $V(F') = V(F) = [\ell]$ and d, $\varepsilon > 0$, if

$$N^*_{F',F}(H_n) = d^{e(F')}(1-d)^{e(F)-e(F')} n^\ell \pm \varepsilon n^\ell\,,$$

where $N^*_{F',F}(H_n)$ denotes the number of labeled, induced copies of F' with respect to F in H_n, i.e., $N^*_{F',F}(H_n)$ is the number of injective mappings $\varphi\colon V(F) \to V(H_n)$ such that for all edges e of the superhypergraph F we have $\varphi(e) \in E(H_n)$ if and only if e is an edge of the subhypergraph F'.

The notion of induced copies with respect to a linear superhypergraph F may look a bit artificial. But it generalizes the usual notion of induced graphs in the case of graphs, as may be seen by setting $F = K_\ell$ to be the complete graph on the same vertex set. We will show that ICL_d is equivalent to DISC_d for k-uniform hypergraphs.

Extension of P_2

Next we focus on a generalization of P_2. For that we need to identify a k-uniform hypergraph which in some sense allows us to reverse the implication from (3.1). Note that there are k-uniform hypergraphs O known, which have the following property: if O appears asymptotically in the "right" frequency in H_n, then H_n must satisfy DISC_d. However, to our knowledge all known k-uniform hypergraphs O with this property are non-linear and, as shown for example in [KNRS10] and discussed above and in Section 1.2.1 of Chapter 1, $\mathrm{DISC}_d(\delta)$ never yields the "right" frequency for any non-linear k-uniform hypergraph O. Below we will give another characterization of the hypergraph M_k, or M for short, that we introduced in Chapter 1. For the proof that both characterizations are equivalent see Lemma 3.1, directly after the discussion of property P_7 below. Recall that M is a linear k-uniform hypergraph M with the same property as C_4 in P_2, i.e., M plays the role of C_4 for $k \geq 3$. (In fact, for $k = 2$ the graph M is equal to C_4.)

For a k-partite k-uniform hypergraph A with vertex classes $X_1, \ldots, X_k$ and $i \in [k]$ we define the *doubling* $\mathrm{db}_i(A)$ *of* A *around class* X_i to be the k-uniform hypergraph obtained from A by taking two disjoint copies of A and identifying the vertices of X_i. More formally, $\mathrm{db}_i(A)$ is the k-partite k-uniform hypergraph with vertex classes $Y_1, \ldots, Y_k$, where $Y_i = X_i$ and for $j \neq i$ we have $Y_j = X_j \dot\cup \widetilde{X}_j$ with $\widetilde{X}_j = \{\widetilde{x} \mid x \in X_j\}$. Thus $\widetilde{x}$ denotes the copy of x. Moreover, the edge set of $\mathrm{db}_i(A)$ is given by

$$E(\mathrm{db}_i(A)) = E(A) \dot\cup \{\{\widetilde{x}_1, \ldots, \widetilde{x}_{i-1}, x_i, \widetilde{x}_{i+1}, \ldots, \widetilde{x}_k\} \colon \{x_1, x_2, \ldots, x_k\} \in E(A)\}.$$

For the construction of the k-uniform hypergraph M we will start with a single hyperedge K_k, which can be seen as a k-partite k-uniform hypergraph with partition classes of size 1, and iteratively *double* this k-uniform hypergraph around the partition classes. More precisely,

$$M = \mathrm{db}_k(\mathrm{db}_{k-1}(\ldots \mathrm{db}_1(K_k)\ldots))\,.$$

More generally, set

$$M_0 = K_k \quad \text{and} \quad M_j = \mathrm{db}_j(M_{j-1}) \quad \text{for } j = 1, \ldots, k,$$

so that $M = M_k$. We observe that for every $j = 0, \ldots, k$ we have

$$|V(M_j)| = j2^{j-1} + (k-j)2^j \quad \text{and} \quad |E(M_j)| = 2^j\,.$$

Moreover, for the vertex partition $X_1 \dot\cup \ldots \dot\cup X_k$ of M_j we have

$$|X_1| = \ldots = |X_j| = 2^{j-1} \quad \text{and} \quad |X_{j+1}| = \ldots = |X_k| = 2^j.$$

As already mentioned for graphs ($k = 2$) the corresponding graph M is C_4 and for $k \geq 3$ the k-uniform hypergraph M will turn out to be the "right" generalization for our purposes. In fact, it follows from the Cauchy-Schwarz inequality that if H_n contains at least $\alpha n^{|V(A)|}$ labeled copies of some given k-partite k-uniform hypergraph A, then H_n contains at least $(\alpha^2 - o(1))n^{|V(\mathrm{db}_i(A))|}$ labeled copies of $\mathrm{db}_i(A)$. Consequently, every k-uniform hypergraph H_n with at least $d\binom{n}{k} + o(n^k)$ edges contains at least $(d^{2^k} - o(1))n^{k2^{k-1}}$ labeled copies of M. Hence, the random k-uniform hypergraph of density d contains approximately the minimum number of copies of M and as we will see k-uniform hypergraphs H_n having $N_M(H_n)$ close to the minimum number will satisfy DISC_d. More precisely, we will show that MIN_d is another property equivalent to DISC_d (see Theorem 1.3), and recall that MIN_d was defined as follows.

$\mathrm{MIN}_d(\varepsilon)$ We say a k-uniform hypergraph H_n on n vertices has $\mathrm{MIN}_d(\varepsilon)$ for d, $\varepsilon > 0$, if

$$e(H_n) \geq d\tbinom{n}{k} - \varepsilon n^k \quad \text{and} \quad N_M(H_n) \leq d^{2^k} n^{k2^{k-1}} + \varepsilon n^{k2^{k-1}}.$$

We did not find any interesting generalization of property P_3 from Theorem 1.1 to k-uniform hypergraphs for $k \geq 3$. Moreover, the extension property P_4 in this work is DISC_d and the generalization of P_5 is straightforward (and the implication $P_5 \Rightarrow P_4$ could be proved along the lines of [Yus08]). Hence, we continue with the discussion of properties P_6 and P_7.

Extension of P_6

The property P_6 is closely related to the appearance of subgraphs of C_4, as shown in [CG91]. For completeness we explain this relation below. More precisely, for a graph G_n let $\mathrm{EVEN}_{C_4}(G_n)$ be the sum of the number of labeled induced copies of subgraphs of C_4 with an even number of edges, i.e.,

$$\mathrm{EVEN}_{C_4}(G_n) = N^*_{\varnothing,C_4}(G_n) + 4N^*_{P_2,C_4}(G_n) + 2N^*_{2K_2,C_4}(G_n) + N^*_{C_4,C_4}(G_n)\,,$$

where $\varnothing$ is the subgraph of C_4 without any edges, P_i is the path with i edges, and $2K_2$ is a matching consisting of two edges. Note, that there are four different ways to select a path of length two within a C_4 and there are two different ways to fix a matching of size two in any given C_4, while there is only one way to fix a C_4 or an "empty C_4" within a cycle of length four. Similarly, set

$$\mathrm{ODD}_{C_4}(G_n) = 4N^*_{P_1,C_4}(G_n) + 4N^*_{P_3,C_4}(G_n)\,.$$

We can rewrite $\mathrm{ODD}_{C_4}(G_n)$ and $\mathrm{EVEN}_{C_4}(G_n)$ in terms of $s(u,v)$ (cf. P_6 in Theorem 1.1) as follows

$$\mathrm{EVEN}_{C_4}(G_n) = \sum_{u,v\in V} \Big(s(u,v)^2 + (n - s(u,v))^2\Big) + o(n^4)$$

and

$$\mathrm{ODD}_{C_4}(G_n) = 2\sum_{u,v\in V}\Big(s(u,v)(n-s(u,v)\Big) + o(n^4)\,.$$

Hence, property P_6 is, due to the Cauchy-Schwarz inequality, equivalent to the following property.

P_6' : $|\mathrm{EVEN}_{C_4}(G_n) - \mathrm{ODD}_{C_4}(G_n)| = \sum_{u,v\in V}(2s(u,v)-n)^2 = o(n^4)$.

For the extension of P_6' to k-uniform hypergraphs, we replace C_4 by M from property MIN_d and in order to deal with arbitrary densities $d>0$ we need a different weight function for the subgraphs of M. For a k-uniform hypergraph H_n and $1\geq d>0$ we define a weight function $w\colon [V(H_n)]^k \to [-1,1]$ and set for $e\in[V(H_n)]^k$

$$w(e) = \begin{cases} 1-d & \text{if } e\in E(H_n) \\ -d & \text{if } e\notin E(H_n)\,. \end{cases}$$

For a labeled copy $\widetilde{A}$ of a given k-uniform hypergraph A in the complete k-uniform hypergraph on $V(H_n)$ we set

$$w(\widetilde{A}) = \prod_{e\in E(\widetilde{A})} w(e)\,.$$

It is easy to check that for a graph G_n and $d=1/2$ we have

$$|\mathrm{EVEN}_{C_4}(G_n) - \mathrm{ODD}_{C_4}(G_n)| = 16\left|\textstyle\sum_{\widetilde{C}_4} w(\widetilde{C}_4)\right| + o(n^4)\,,$$

where the sum runs over all labeled copies $\widetilde{C}_4$ of C_4 in the complete graph on $V(G_n)$.

With this in mind, we define the generalization of P_6 as follows, which may be viewed as a weighted form of MIN_d.

$\mathrm{DEV}_d(\varepsilon)$ We say a k-uniform hypergraph H_n on n vertices has $\mathrm{DEV}_d(\varepsilon)$ for d, $\varepsilon>0$, if

$$\left|\textstyle\sum_{\widetilde{M}} w(\widetilde{M})\right| \leq \varepsilon n^{k2^{k-1}}$$

where the sum runs over all labeled copies $\widetilde{M}$ of M in the complete k-uniform hypergraph on $V(H_n)$.

Again Theorem 1.3 will show that DEV_d is equivalent to DISC_d.

Extension of P_7

The last property we consider here is P_7. Roughly speaking, P_7 asserts that most pairs of vertices of G_n have approximately $n/4$ neighbors and this implies, on the one hand, that the number of labeled C_4's in G_n is close to $n^4/16$, while, on the other hand, for most vertices v the number of labeled C_4's containing v satisfies $\sum_{w\in V}(\mathrm{codeg}(v,w))^2 \sim n\times(n/4)^2$ as well as $\sum_{u,u'\in N(v)}\mathrm{codeg}(u,u') \sim (\deg(v))^2(n/4)$, which yields $\deg(v)\sim n/2$

for almost all vertices v. Consequently, P_7 implies P_2 and the reverse implication follows from the Cauchy-Schwarz inequality. From this point of view the obvious generalization of P_7 concerns the number of labeled copies of M_{k-1} attached to a fixed, labeled set of 2^{k-1} vertices. We now make this precise.

Let H_n be a k-uniform hypergraph on n vertices. Let X_k be the (unique) largest vertex class of M_{k-1} and, for $q = 2^{k-1}$, let $x_1, \ldots, x_q$ be an arbitrary labeling of the vertices of X_k. For an ordered set $\boldsymbol{u} = (u_1, \ldots, u_q)$ of q vertices in $V(H_n)$, we denote by $\mathrm{ext}(M_{k-1}, H_n, \boldsymbol{u})$ the number of copies of M_{k-1} in H_n extending $\boldsymbol{u}$ in a canonical way, i.e., $\mathrm{ext}(M_{k-1}, H_n, \boldsymbol{u})$ is the number of injective, edge preserving mappings $\varphi\colon V(M_{k-1}) \to V(H_n)$ with $\varphi(x_i) = u_i$ for $i = 1, \ldots, q$. The generalization of P_7 then reads as follows.

$\mathrm{MDEG}_d(\varepsilon)$ We say a k-uniform hypergraph H_n on n vertices has $\mathrm{MDEG}_d(\varepsilon)$ for d, $\varepsilon > 0$, if

$$\textstyle\sum_{\boldsymbol{u}} \left|\mathrm{ext}(M_{k-1}, H_n, \boldsymbol{u}) - d^{2^{k-1}} n^{(k-1)2^{k-2}}\right| \leq \varepsilon n^{(k+1)2^{k-2}}$$

where the sum runs over all ordered 2^{k-1}-element subsets $\boldsymbol{u}$ in $V(H_n)$.

In the proof of Theorem 1.3 we will use (3.1) which was proved in [KNRS10]. We will include a direct proof of the implication from DEV_d to CL_d and another proof that DISC_d implies CL_d in Section 3.2.5.

Equivalent characterizations of M_k

It was mentioned earlier that M_k has two characterizations: a constructive one associated with the concept of doubling and a combinatorial one coming from the k-dimensional hypercube Q_k.

Lemma 3.1. *Let M' be the k-uniform hypergraph obtained from the k-dimensional hypercube Q_k by identifying the edges of Q_k with the vertex set of M' and any hyperedge of M' to consist of those edges of Q_k that meet in a vertex. Then M' is isomorphic to the hypergraph*

$$M_k := \mathrm{db}_k(\mathrm{db}_{k-1}(\ldots \mathrm{db}_1(K_k)\ldots)) .$$

Moreover, the vertex class X_i of M_k corresponds to all edges $\{u, v\}$ of Q_k, where u and v differ exactly in the i-th coordinate.

Proof. We show the equivalence by induction on k. For this, we generalize the notation for M_k by letting $M_j^{(i)}$ to be the i-uniform hypergraph after j doubling steps of the single hyperedge $K_i^{(i)}$. For $k = 2$ we deal with the cycle C_4, which is isomorphic to Q_2, so the induction base is shown.

Suppose we have an isomorphism φ from $M_{k-1}^{(k-1)}$ to Q_{k-1}. Recall, that every vertex of $M_{k-1}^{(k-1)}$ corresponds to an edge $(u, v) \in E(Q_{k-1})$ with $u, v \in \{0, 1\}^{k-1}$ and u, v differ in exactly one coordinate. Moreover, every hyperedge in $M_{k-1}^{(k-1)}$ is of type $\{\{u_1, v\}, \{u_2, v\}, \ldots, \{u_{k-1}, v\}\}$, and u_i's differ in different coordinates from v with u_i,

$v \in \{0,1\}^{k-1}$. We embed the hypergraph $M_{k-1}^{(k)}$ in Q_k as follows. First we identify $M_{k-1}^{(k-1)}$ with Q_{k-1} via the isomorphism φ, which is given to us by the induction hypothesis. Then, every $\{u,v\}$ is enlarged to $\{(u,0),(v,0)\}$, and the hyperedge $\{\{u_1,v\},\{u_2,v\},\ldots,\{u_{k-1},v\}\}$ becomes

$$\{\{(u_1,0),(v,0)\},\{(u_2,0),(v,0)\},\ldots,\{(u_{k-1},0),(v,0)\},\{(v,1),(v,0)\}\}.$$

Note that v runs over all elements of $\{0,1\}^{k-1}$. And the edges of Q_k of the type

$$\{(v,0),(v,1)\}\,,\text{where}\, v \in \{0,1\}^{k-1}$$

constitute the X_k-class of $M_{k-1}^{(k)}$ (and also of $M_k^{(k)}$). Finally, to obtain the hypergraph $M = M_k^{(k)}$, we fix the last class and double the first $(k-1)$ classes together with the hyperedges. Note now that so far we dealt with hyperedges of the form (recalling again that u_i and v are adjacent vertices in Q_{k-1})

$$\{\{(u_1,0),(v,0)\},\{(u_2,0),(v,0)\},\ldots,\{(u_{k-1},0),(v,0)\},\{(v,1),(v,0)\}\}.$$

Any hyperedge of the form above gives rise to the hyperedge

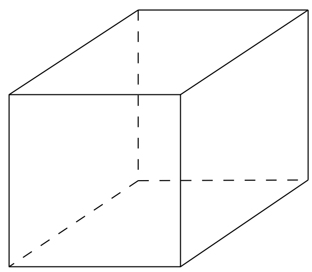

Figure 3.1: The 3-dimensional hypercube Q_3

$$\{\{(u_1,1),(v,1)\},\{(u_2,1),(v,1)\},\ldots,\{(u_{k-1},1),(v,1)\},\{(v,1),(v,0)\}\}. \tag{3.2}$$

and the vertices (of M) of the type $\{(u,1),(v,1)\}$ with $u,v \in \{0,1\}^{k-1}$(and u,v differ in exactly one coordinate) are genuinly new. Moreover these hyperedges (3.2) are created when we perform the operation of doubling of $M_k^{(k-1)}$ to get $M = M_k^{(k)}$. $\square$

Remark 3.2. This combinatorial characterization of M_k through Q_k allows us to uncover at least simple isomorphisms of M_k very quickly. Let $X_1,\ldots,X_k$ denote the partition classes of M_k, and note that, by our discussion, every class X_i of M corresponds to those edges of Q_k that connect vertices by changing the coordinate i. Then, for every $i,j \in [k], i \neq j$ there exists an isomorphism φ of M_k, i.e. $\varphi\colon V(M_k) \to V(M_k)$ such that $\varphi(X_i) = X_j$, $\varphi(X_j) = X_i$ and $\varphi(X_s) = \varphi(X_s)$ for every $s \neq i,j$. Geometrically, φ can be interpreted as an appropriate rotation of the hypercube Q_k by rotating x_i-axis to the x_j-axis. More generally, for any $j \in [k-1]$ there is an isomorphism φ_j of M_{j+1},

$\varphi_j \colon V(M_{j+1}) \to V(M_{j+1})$, such that $\varphi_j(X_{j+1}(M_{j+1})) = X_j(M_{j+1})$, $\varphi_j(X_j(M_{j+1})) = X_{j+1}(M_{j+1})$ and $\varphi_j(X_s(M_{j+1})) = X_s(M_{j+1})$ for every $s \neq j, j+1$. This fact will be used later in the proof MDEG $\Longrightarrow$ MIN.

3.1.2 Forcing pairs for graphs

Theorem 1.3, although a result about k-uniform hypergraphs, has an interesting consequence for graphs. Recall that we defined forcing pairs as those pairs of graphs, such that the lower bound on the occurrences on the one graph together with the upper bound on the other imply quasi-randomness of a graph. Defining $M(k)$ to be the line graph of the graph of the k-dimensional hypercube Q_k, we show below using Theorem 1.3 combined with Theorem 1.2, that $(K_k, M(k))$ is a forcing pair.

Proof of Corollary 1.4. From the given graph G we construct a k-uniform hypergraph $H = H(G)$, where the hyperedges of H correspond to the cliques K_k of G. Therefore we have a one-to-one correspondence between the hyperedges of H and the K_k's of G, as well as, between the copies of M_k in H and the copies of $M(k)$ in G. Hence, the assumption on G implies that H satisfies $\mathrm{MIN}_{d'}$ for k-uniform hypergraphs for $d' = d^{\binom{k}{2}}$ and from Theorem 1.3 we infer that H satisfies $\mathrm{DISC}_{d'}(\varepsilon')$ for k-uniform hypergraphs for some $\varepsilon' = \varepsilon'(\varepsilon)$ with $\varepsilon' \to 0$ as $\varepsilon \to 0$. But $\mathrm{DISC}_{d'}(\varepsilon')$ for H implies that the assumption of Theorem 1.2 for the graphs $F = K_k$ and G are met and, hence, Theorem 1.2 yields that G satisfies $\mathrm{DISC}_d(\delta)$ for graphs for some $\delta = \delta(\varepsilon')$. $\square$

3.1.3 Hereditary subgraphs properties

From Theorem 1.3 we know that k-uniform hypergraphs containing the "right" number of copies of M are quasi-random. However, note that for characterizing quasi-randomness the linear k-uniform hypergraph M cannot be replaced by an arbitrary (linear) k-uniform hypergraph. For example, in the case of graphs, the C_4 in P_2 cannot be replaced by a triangle, as the following example from [CGW89] shows: partition the vertex set $V(G_n)$ in four sets $X_1 \dot\cup X_2 \dot\cup X_3 \dot\cup X_4 = V(G_n)$ as equal as possible and add the edges of the complete graph on X_1, of the complete graph on X_2, of the complete bipartite graph with vertex classes X_3 and X_4, and of the random bipartite graph of density $1/2$ with vertex classes $X_1 \dot\cup X_2$ and $X_3 \dot\cup X_4$. Simple calculations show, that G_n defined this way has density $1/2 + o(1)$ and contains $n^3/8 + o(n^3)$ labeled triangles. On the other hand, G_n is not quasi-random, as it obviously violates P_4. Moreover, due to Theorem 1.1, a quasi-random graph must be *hereditarily* quasi-random, since if G_n satisfies P_4, then induced subgraphs $G_n[U]$ for large subsets also satisfy P_4 (with a bigger error). Consequently, any property equivalent to P_4 must directly apply to induced subgraphs of linear sized subsets. (It is not obvious that all the properties in Theorem 1.1 indeed have this quality, but e.g. due to Theorem 1.1 it follows.) Returning to the example of triangles, we note that the "counterexample" shows that there are graphs which have globally the "right" number of triangles, but there are large subsets on which the number of triangles is wrong, e.g. $G_n[X_1]$ contains too many (more than $(n/4)^3/8$) triangles. In order to rule out this phenomenon Simonovits and Sós suggested a notion of hereditary properties and in [SS97] they showed that a graph G with density d is quasi-random if and only if every induced subgraph of G contains the right number of copies of a fixed graph F (see Theorem 1.2). This result has been extended to the case of induced copies of F by Simonovits and Sós [SS03] and by Shapira and Yuster [SY08]. We will continue this line of research and introduce hereditary properties for k-uniform hypergraphs, which are equivalent to DISC_d.

Let H_n be a k-uniform hypergraph on n vertices and let F be a k-uniform hypergraph with vertex set $[\ell] = \{1, \ldots, \ell\}$. Recall that for pairwise disjoint sets $U_1, \ldots, U_\ell \subseteq V(H_n)$, $N_F(U_1, \ldots, U_\ell)$ denotes the number of partite-isomorphic, copies of F in H_n, i.e., the number of ℓ-tuples $(h_1, \ldots, h_\ell)$ with $h_1 \in U_1, \ldots, h_\ell \in U_\ell$ such that $\{h_{i_1}, \ldots, h_{i_k}\}$ is an edge in H_n if $\{i_1, \ldots, i_k\}$ is an edge in F. We define the following properties and show that they are equivalent to DISC_d.

$\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}(\varepsilon)$ We say a k-uniform hypergraph H_n on n vertices has $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}(\varepsilon)$ for a linear k-uniform hypergraph F with $V(F) = [\ell]$, a vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \in (0,1)^\ell$ with $\sum_{i=1}^{\ell} \alpha_i < 1$, and d, $\varepsilon > 0$, if for all choices of pairwise disjoint subsets $U_1, \ldots, U_\ell \subset V(H_n)$ with $|U_i| = \lfloor \alpha_i n \rfloor$ for all $i \in [\ell]$ we have

$$N_F(U_1, \ldots, U_\ell) = d^{e(F)} \prod_{i \in [\ell]} |U_i| \pm \varepsilon n^\ell .$$

$\mathrm{HCL}_{d,F}(\varepsilon)$ We say a k-uniform hypergraph H_n on n vertices has $\mathrm{HCL}_{d,F}(\varepsilon)$ for a linear k-uniform hypergraph F with $V(F) = [\ell]$ and d, $\varepsilon > 0$, if H_n satisfies $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}(\varepsilon)$ for every vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \in (0,1)^\ell$ with $\sum_{i=1}^{\ell} \alpha_i < 1$.

Theorem 3.3. *For every integer $k \geq 2$, every linear k-uniform hypergraph F with at least one edge and $V(F) = [\ell]$, every $d > 0$, and every vector $\boldsymbol{\alpha} \in (0,1)^\ell$ with $\sum_{i=1}^{\ell} \alpha_i < 1$ the properties DISC_d, $\mathrm{HCL}_{d,F}$, and $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}$ are equivalent.*

We prove Theorem 3.3 in Section 3.3. We would also like to mention that the property $\mathrm{HCL}_{d,F}$ can be weakened in the graph case. In fact, Theorem 1.2 shows that it suffices to ensure approximately the right number of copies of the fixed graph F in every subset $U \subseteq V(G_n)$ of the vertices of G_n to make G_n quasi-random. We, however, need the assumption for all partitions of U into ℓ classes. It seems quite plausible that this stronger looking assumption is not needed and, in fact, for k-uniform hypergraphs this was proved recently by Dellamonica and Rödl [DR].

3.1.4 Partite versions of DISC

Property P_4 of Theorem 1.1 has a very natural bipartite version, stating that the number of edges between two subsets is close to half of all possible edges between those sets. More precisely, we may consider the following property.

P_4' $e(U,W) = |U||W|/2 + o(n^2)$ for all pairwise disjoint subsets U, $W \subseteq V(G_n)$, where $e(U,W)$ denotes the number of edges with one vertex in U and one vertex in W.

It is well known that in fact P_4 and P_4' are equivalent. For example P_4 implies P_4' due to the identity $e(U,W) = e(U \cup W) - e(U) - e(W)$, while P_4 follows from P_4' by considering $e(U', W')$ for a random partition $U = U' \dot\cup W'$ of a given set U into classes of size $|U|/2$.

Below we introduce several partite variants of DISC_d for k-uniform hypergraphs, which will turn out to be equivalent. We start with some definitions. For integers $1 \leq \ell \leq k$ we call $\tau\colon [\ell] \to [k]$ an *(ℓ, k)-function* if $\sum_{i \in [\ell]} \tau(i) = k$. The set of all (ℓ, k)-functions will be denoted by $T(\ell, k)$. For a fixed $\tau \in T(\ell, k)$ and ℓ pairwise disjoint sets $U_1, \dots, U_\ell \subset V$ of some vertex set V we say a k-set $K \in [V]^k$ has *type τ (with respect to $(U_1, \dots, U_\ell)$)*, if $|K \cap U_i| = \tau(i)$ for all $i \in [\ell]$. The family of all k-sets having type τ is denoted by

$$\mathrm{Vol}_\tau(U_1, \dots, U_\ell) = \{K \in [V]^k \colon K \text{ has type } \tau\}$$

and let $\mathrm{vol}_\tau(U_1, \dots, U_\ell) = |\mathrm{Vol}_\tau(U_1, \dots, U_\ell)| = \prod_{i \in [\ell]} \binom{|U_i|}{\tau(i)}$.

Alternatively $\mathrm{Vol}_\tau(U_1, \dots, U_\ell)$ can be considered the complete k-uniform hypergraph with respect to type τ. The actual edges of a k-uniform hypergraph H_n with vertex set V of type τ with respect to $(U_1, \dots, U_\ell)$ will be denoted by

$$E_\tau(U_1, \dots, U_\ell) = E(H_n) \cap \mathrm{Vol}_\tau(U_1, \dots, U_\ell)$$

and we set $e_\tau(U_1, \dots, U_\ell) = |E_\tau(U_1, \dots, U_\ell)|$. Note that for $k = 2$ and $\ell = 1, 2$ there exists only one (ℓ, k)-function and edges of the corresponding type are considered in P_4 ($\ell = 1$) and in P_4' ($\ell = 2$). For general $k \geq 2$ we define the following property.

$\mathrm{DISC}_{d,\tau}(\varepsilon)$ We say a k-uniform hypergraph H_n on n vertices has $\mathrm{DISC}_{d,\tau}(\varepsilon)$ for some (ℓ, k)-function τ, and d, $\varepsilon > 0$, if

$$e_\tau(U_1, \dots, U_\ell) = d \cdot \mathrm{vol}_\tau(U_1, \dots, U_\ell) \pm \varepsilon n^k$$

for all pairwise disjoint subsets $U_1, \dots, U_\ell \subseteq V(H_n)$.

Next, we define the notion of the ℓ-partite k-uniform subhypergraph with respect to the pairwise disjoint sets $U_1, \dots, U_\ell \subset V(H_n)$. The edge set of the complete ℓ-partite k-uniform hypergraph with respect to the classes $U_1, \dots, U_\ell$ is given by

$$\mathrm{Vol}(U_1, \dots, U_\ell) = \bigcup_{\tau \in T(\ell,k)} \mathrm{Vol}_\tau(U_1, \dots, U_\ell) \tag{3.3}$$

and the actual edge set of the ℓ-partite k-uniform subhypergraph on $U_1, \dots, U_\ell$ is

$$E(U_1, \dots, U_\ell) = E(H_n) \cap \mathrm{Vol}(U_1, \dots, U_\ell). \tag{3.4}$$

Finally, we consider the following notion of uniform edge distribution.

$\mathrm{DISC}_{d,\ell}(\varepsilon)$ We say a k-uniform hypergraph H_n on n vertices has $\mathrm{DISC}_{d,\ell}(\varepsilon)$ for some positive integer $\ell \leq k$, and d, $\varepsilon > 0$, if

$$e(U_1, \dots, U_\ell) = d \cdot \mathrm{vol}(U_1, \dots, U_\ell) \pm \varepsilon n^k$$

for all pairwise disjoint subsets $U_1, \ldots, U_\ell \subseteq V(H_n)$.

Note that for arbitrary k the properties DISC_d, $\mathrm{DISC}_{d,1}$, and $\mathrm{DISC}_{d,(1)}$ are the same and $\mathrm{DISC}_{d,k}$ and $\mathrm{DISC}_{d,(1,\ldots,1)}$ are the same. Moreover, for $k = 2$ these two properties are equivalent. The following result states that in fact any version of DISC defined above is equivalent to any other.

Theorem 3.4. *For all integers ℓ and k with $1 \leq \ell \leq k$, every fixed (ℓ, k)-function τ, and every $d > 0$ the properties DISC_d, $\mathrm{DISC}_{d,\ell}$, and $\mathrm{DISC}_{d,\tau}$ are equivalent.*

Finally let us consider the following property which might be seen as a generalization of $\mathrm{DISC}_{d,k}$ and a special case of $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}$, referred to as cut property:

$\mathrm{CUT}_{d,\ell,\boldsymbol{\alpha}}(\varepsilon)$ We say a k-uniform hypergraph H_n on n vertices has $\mathrm{CUT}_{d,\ell,\boldsymbol{\alpha}}(\varepsilon)$ for a vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_\ell) \in (0,1)^\ell$ with $\sum_{i=1}^{\ell} \alpha_i = 1$, $\ell \geq k$ and d, $\varepsilon > 0$, if for all choices of pairwise disjoint subsets $U_1, \ldots, U_\ell \subset V(H_n)$ with $|U_i| = \lfloor \alpha_i n \rfloor$ for all $i \in [\ell]$ we have

$$e(U_1, \ldots, U_\ell) = d \cdot \textstyle\sum_{S \subseteq [\ell], |S| = k} \prod_{i \in S} |U_i| \pm \varepsilon n^k,$$

where $e(U_1, \ldots, U_\ell)$ denotes the number of crossing hyperedges of H_n in the partition $U_1 \dot\cup \ldots \dot\cup U_\ell$.

If $\boldsymbol{\alpha} = (1/\ell, \ldots, 1/\ell)$ then this property is called *balanced*. Very recently Shapira and Yuster [SYb], generalizing a result of Chung and Graham [CG92a] to k-uniform hypergraphs showed that $\mathrm{CUT}_{d,\ell,\boldsymbol{\alpha}}(\varepsilon)$ is a quasi-random property if and only if it is not balanced. Moreover, they characterized those hypergraphs that satisfy a balanced cut property.

3.2 Proof of Theorem 1.3

In this section we present the proof of Theorem 1.3. We have to show that for every $k \geq 2$ and every $d > 0$ the properties DISC_d, CL_d, ICL_d, MIN_d, DEV_d, and MDEG_d are equivalent. As already noted in (3.1) it was shown in [KNRS10] that DISC_d implies CL_d. In Section 3.2.1 we will show the following obvious implications: $\mathrm{CL}_d \Longrightarrow \mathrm{MIN}_d$ (Fact 3.5), $\mathrm{CL}_d \Longrightarrow \mathrm{ICL}_d$ (Fact 3.6) and $\mathrm{ICL}_d \Longrightarrow \mathrm{DEV}_d$ (Fact 3.7). The proofs of the main implications $\mathrm{MIN}_d \Longrightarrow \mathrm{DISC}_d$ (Lemma 3.8) and $\mathrm{DEV}_d \Longrightarrow \mathrm{DISC}_d$ (Lemma 3.11) will be given in Sections 3.2.2 and 3.2.3. Finally, we prove the equivalence of MDEG_d and MIN_d in Section 3.2.4 (see Lemma 3.12), which concludes the proof of Theorem 1.3.

Additionally, in Section 3.2.5 we verify a more direct proof of the implication from DEV_d to CL_d by introducing another property FDISC_d.

3.2.1 Simple facts

In this section we verify the above "obvious" implications. The first implication, $\mathrm{CL}_d \Rightarrow \mathrm{MIN}_d$, follows from the definition that a sequence $(H_n)_{n \in \mathbb{N}}$ satisfies CL_d if for every linear

k-uniform hypergraph F and every $\varepsilon > 0$ all but finitely many k-uniform hypergraphs H_n of the sequence satisfy $\mathrm{CL}_d(F, \varepsilon)$.

Fact 3.5. *For every integer $k \geq 2$, every $d > 0$, and every $\varepsilon > 0$ there exists n_0 such that the following is true. If H is a k-uniform hypergraph that satisfies $\mathrm{CL}_d(K_k, \varepsilon/2)$ and $\mathrm{CL}_d(M, \varepsilon)$, then H satisfies $\mathrm{MIN}_d(\varepsilon)$.*

Proof. Clearly, satisfying $\mathrm{CL}_d(K_k, \varepsilon/2)$ implies $e(H_n) \geq d\binom{n}{k} - \varepsilon n^k$ for sufficiently large n and satisfying $\mathrm{CL}_d(M, \varepsilon)$ yields $N_M(H) \leq d^{|E(M)|} n^{|V(M)|} + \varepsilon n^{|V(M)|}$, which gives $\mathrm{MIN}_d(\varepsilon)$. $\square$

A standard argument using the principle of inclusion and exclusion yields the implication from CL_d to ICL_d.

Fact 3.6. *For every integer $k \geq 2$, every $d > 0$, all linear k-uniform hypergraphs $F' \subseteq F$ with $V(F') = V(F) = [\ell]$ for some integer ℓ, and every $\varepsilon > 0$, there exists $\delta > 0$ such that the following is true. If H is a k-uniform hypergraph that satisfies $\mathrm{CL}_d(\hat{F}, \delta)$ for every k-uniform hypergraph $\hat{F}$ with $F' \subseteq \hat{F} \subseteq F$, then H satisfies $\mathrm{ICL}_d(F', F, \varepsilon)$.*

Proof. Let $\delta = \varepsilon/2^{e(F)-e(F')}$ and H be a k-uniform hypergraph on n vertices. Note that by the principle of inclusion and exclusion we have

$$N^*_{F',F}(H) = \sum_{F' \subseteq \hat{F} \subseteq F} (-1)^{e(\hat{F})-e(F')} N_{\hat{F}}(H)\,.$$

Since H satisfies $\mathrm{CL}_d(\hat{F}, \delta)$ for every k-uniform hypergraph $\hat{F}$ with $F' \subseteq \hat{F} \subseteq F$ we obtain

$$N^*_{F',F}(H) = d^{e(F')}(1-d)^{e(F)-e(F')} n^\ell \pm 2^{e(F)-e(F')} \delta n^\ell\,,$$

which shows that H satisfies $\mathrm{ICL}_d(F', F, \varepsilon)$. $\square$

We close this section by observing that ICL_d implies DEV_d.

Fact 3.7. *For every integer $k \geq 2$, every $d > 0$, and every $\varepsilon > 0$, there exists $\delta > 0$ such that the following is true. If H is a k-uniform hypergraph that satisfies $\mathrm{ICL}_d(M', M, \delta)$ for every k-uniform hypergraph $M' \subseteq M$, then H satisfies $\mathrm{DEV}_d(\varepsilon)$.*

Proof. Set $\delta = \varepsilon/2^{2^k}$. Let H be a k-uniform hypergraph on n vertices with vertex set $V = V(H)$ satisfying $\mathrm{ICL}_d(M', M, \delta)$ for every $M' \subseteq M$. Recall that the edge weights w of the complete k-uniform hypergraph K_V on V are $1-d$ for edges of H and $-d$ for edges of the complement of H. Moreover, $w(\widetilde{A})$ for subgraph $\widetilde{A} \subseteq K_V$ is $\prod_{e \in E(\widetilde{A})} w(e)$. Summing over all copies $\widetilde{M}$ of M in K_V we obtain

$$\sum_{\widetilde{M}} w(\widetilde{M}) = \sum_{M' \subseteq M} (1-d)^{e(M')}(-d)^{2^k - e(M')} N^*_{M',M}(H)\,.$$

Applying the assumption that H satisfies $\mathrm{ICL}_d(M', M, \delta)$ for all k-uniform hypergraphs $M' \subseteq M$ we get

$$\begin{aligned}\sum_{\widetilde{M}} w(\widetilde{M}) &= \sum_{M' \subseteq M} (1-d)^{e(M')}(-d)^{2^k - e(M')}\Big(d^{e(M')}(1-d)^{2^k-e(M')} \pm \delta\Big) n^{|V(M)|} \\ &= \sum_{j=0}^{2^k} \binom{2^k}{j} \Big(d(1-d)\Big)^j \Big((-d)(1-d)\Big)^{2^k - j} n^{|V(M)|} \pm 2^{2^k} \delta n^{|V(M)|}\,.\end{aligned}$$

Consequently, the binomial theorem and the choice of δ yields DEV_d,

$$\Big|\textstyle\sum_{\widetilde{M}} w(\widetilde{M})\Big| \leq \varepsilon n^{|V(M)|}\,.$$

□

3.2.2 MIN_d implies DISC_d

In this section we focus on one of the central implications of Theorem 1.3 and prove the following lemma, which asserts that MIN_d implies DISC_d.

Lemma 3.8. *For every integer $k \geq 2$, every $d > 0$, and every $\varepsilon > 0$, there exist $\delta > 0$ and n_0 such that the following is true. If H is a k-uniform hypergraph on $n \geq n_0$ vertices that satisfies $\mathrm{MIN}_d(\delta)$, then H satisfies $\mathrm{DISC}_d(\varepsilon)$.*

Before we prove Lemma 3.8 we introduce a bit of notation, which will be also useful for the proof of Lemma 3.11. It will be convenient to consider the number of homomorphisms from certain k-uniform hypergraphs A to the k-uniform hypergraph H, instead of the number of labeled copies of A in H. Recall that a homomorphism from A to H is a (not necessarily injective) mapping from $V(A)$ to $V(H)$ that preserves edges. Note that the difference of the number of homomorphisms and the number of labeled copies of A in H is $o(|V(H)|^{|V(A)|})$, which is inessential for the properties considered in Theorem 1.3.

Let A be a k-partite k-uniform hypergraph given with its partition classes $X_1, \dots, X_k$ and let $U_1, \dots, U_k$ be (not necessarily pairwise disjoint) subsets of $V(H)$ and set $\mathcal{U} = (U_1, \dots, U_k)$. We denote by $\mathrm{Hom}(A, H, \mathcal{U})$ those homomorphisms φ from A to H that map every X_i into U_i, i.e. $\varphi(X_i) \subseteq U_i$ for all $i \in [k]$. Furthermore, let $\hom(A, H, \mathcal{U}) = |\mathrm{Hom}(A, H, \mathcal{U})|$.

Moreover, let $X_i = \{x_{i,1}, \dots, x_{i,|X_i|}\}$ be a labeling of the vertices of the partition class X_i. Then, for an $|X_i|$-tuple $\boldsymbol{u}_i = (u_1, \dots, u_{|X_i|}) \in U_i^{|X_i|}$ denote by $\mathrm{Hom}(A, H, \mathcal{U}, i, \boldsymbol{u}_i)$ those homomorphisms φ from $\mathrm{Hom}(A, H, \mathcal{U})$, that map the j-th vertex in the ordering of X_i to u_j, i.e., $\varphi(x_{i,j}) = u_j$. Similarly, let $\hom(A, H, \mathcal{U}, i, \boldsymbol{u}_i) = |\mathrm{Hom}(A, H, \mathcal{U}, i, \boldsymbol{u}_i)|$.

The following well known fact (for the proof see, e.g. [ST04]) will be useful for the proof of Lemma 3.8.

Fact 3.9. *For every $\gamma > 0$ there exists $\eta > 0$ such that for all non-negative reals $a_1, \dots, a_N$ and a satisfying $\sum_{i=1}^N a_i \geq (1-\eta)aN$ and $\sum_{i=1}^N a_i^2 \leq (1+\eta)a^2 N$, we have $|\{i \in [N]\colon\ |a - a_i| < \gamma a\}| > (1-\gamma)N$.* □

Proof of Lemma 3.8. We first make a few observations (see Claim 3.10 below). For that let H be a k-uniform hypergraph with vertex set $V = V(H)$ and let $U_1, \ldots, U_k$ be arbitrary, not necessarily disjoint, subsets of V. Set $\mathcal{U} = (U_1, \ldots, U_k)$. For every $j \in [k]$ the Cauchy-Schwarz inequality yields

$$\sum_{\boldsymbol{u}_j \in U_j^{2^{j-1}}} \left(\hom(M_{j-1}, H, \mathcal{U}, j, \boldsymbol{u}_j)\right)^2 \geq \frac{1}{|U_j|^{2^{j-1}}} \left(\sum_{\boldsymbol{u}_j \in U_j^{2^{j-1}}} \hom(M_{j-1}, H, \mathcal{U}, j, \boldsymbol{u}_j) \right)^2 . \tag{3.5}$$

Furthermore note, that $M_j = \mathrm{db}_j(M_{j-1})$, i.e., M_j arises from M_{j-1} by "fixing" the vertices from the j-th partition class of M_{j-1}, denoted by $X_j(M_{j-1})$, and "doubling" all other vertices of M_{j-1} and the corresponding edges. Thus, this definition yields the following identity for every $j \in [k]$.

$$\hom(M_j, H, \mathcal{U}) = \sum_{\boldsymbol{u}_j \in U_j^{2^{j-1}}} \hom(M_j, H, \mathcal{U}, j, \boldsymbol{u}_j) = \sum_{\boldsymbol{u}_j \in U_j^{2^{j-1}}} \left(\hom(M_{j-1}, H, \mathcal{U}, j, \boldsymbol{u}_j)\right)^2. \tag{3.6}$$

Combining (3.5) and (3.6), we get

$$\begin{aligned}\hom(M_j, H, \mathcal{U}) &\overset{(3.6)}{=} \sum_{\boldsymbol{u}_j \in U_j^{2^{j-1}}} \left(\hom(M_{j-1}, H, \mathcal{U}, j, \boldsymbol{u}_j)\right)^2 \\ &\overset{(3.5)}{\geq} \frac{1}{|U_j|^{2^{j-1}}} \left(\sum_{\boldsymbol{u}_j \in U_j^{2^{j-1}}} \hom(M_{j-1}, H, \mathcal{U}, j, \boldsymbol{u}_j) \right)^2 \\ &= \frac{1}{|U_j|^{2^{j-1}}} \left(\hom(M_{j-1}, H, \mathcal{U})\right)^2 .\end{aligned}$$

Iterating the last estimate $j - \ell + 1$ times for some $1 \leq \ell \leq j$ we get the following line of inequalities for every integer r between ℓ and j

$$\begin{aligned}\hom(M_j, H, \mathcal{U}) &= \sum_{\boldsymbol{u}_j \in U_j^{2^{j-1}}} \left(\hom(M_{j-1}, H, \mathcal{U}, j, \boldsymbol{u}_j)\right)^2 \\ &\geq \left(\frac{1}{|U_j|}\right)^{2^{j-1}} \left(\sum_{\boldsymbol{u}_j \in U_j^{2^{j-1}}} \hom(M_{j-1}, H, \mathcal{U}, j, \boldsymbol{u}_j) \right)^2 \\ &\ldots\end{aligned} \tag{3.7}$$

$$\geq \left(\prod_{i=r+1}^{j} \frac{1}{|U_i|}\right)^{2^{j-1}} \left(\sum_{\boldsymbol{u}_r \in U_r^{2^{r-1}}} \big(\hom(M_{r-1}, H, \mathcal{U}, r, \boldsymbol{u}_r)\big)^2\right)^{2^{j-r}} \tag{3.8}$$

$$\geq \left(\prod_{i=r}^{j} \frac{1}{|U_i|}\right)^{2^{j-1}} \left(\sum_{\boldsymbol{u}_r \in U_r^{2^{r-1}}} \hom(M_{r-1}, H, \mathcal{U}, r, \boldsymbol{u}_r)\right)^{2^{j-r+1}} \tag{3.9}$$

$$\dots$$

$$= \left(\prod_{i=\ell}^{j} \frac{1}{|U_i|}\right)^{2^{j-1}} \Big(\hom(M_{\ell-1}, H, \mathcal{U})\Big)^{2^{j-\ell+1}}. \tag{3.10}$$

Combining the last line of inequalities with Fact 3.9 yields the following claim.

Claim 3.10. *For all integers $k \geq j \geq \ell \geq 1$ and every $\gamma_{j,\ell} > 0$ there exists $\eta_{j,\ell} > 0$ such that for all $\mathcal{U} = (U_1, \dots, U_k)$ with $U_i \subseteq V$ the following is true. If*

(*a*) $\hom(M_{\ell-1}, H, \mathcal{U}) \geq (1 - \eta_{j,\ell}) d^{2^{\ell-1}} \prod_{i=1}^{\ell-1} |U_i|^{2^{\ell-2}} \prod_{i=\ell}^{k} |U_i|^{2^{\ell-1}}$ *and*

(*b*) $\hom(M_j, H, \mathcal{U}) \leq (1 + \eta_{j,\ell}) d^{2^j} \prod_{i=1}^{j} |U_i|^{2^{j-1}} \prod_{i=j+1}^{k} |U_i|^{2^j}$

hold, then for every r with $\ell \leq r \leq j$ the following holds. For all but at most $\gamma_{j,\ell}|U_r|^{2^{r-1}}$ tuples $\boldsymbol{u}_r = (u_1, \dots, u_{2^{r-1}})$ from $U_r^{2^{r-1}}$ we have

$$\hom(M_{r-1}, H, \mathcal{U}, r, \boldsymbol{u}_r) = (1 \pm \gamma_{j,\ell}) d^{2^{r-1}} \prod_{i=1}^{r-1} |U_i|^{2^{r-2}} \prod_{i=r+1}^{k} |U_i|^{2^{r-1}}.$$

Proof of Claim 3.10. Note that the assumptions (*a*) and (*b*) of the claim yield a lower bound for the right-hand side of (3.10) and an upper bound for the left-hand side in (3.7). Consequently, for every r between ℓ and j we obtain from (3.8) and (3.9)

$$\sum_{\boldsymbol{u}_r \in U_r^{2^{r-1}}} \big(\hom(M_{r-1}, H, \mathcal{U}, r, \boldsymbol{u}_r)\big)^2 \leq (1 + \eta_{j,\ell})^{1/2^{j-r}} d^{2^r} \prod_{i=1}^{r} |U_i|^{2^{r-1}} \prod_{i=r+1}^{k} |U_i|^{2^r}$$

and

$$\sum_{\boldsymbol{u}_r \in U_r^{2^{r-1}}} \hom(M_{r-1}, H, \mathcal{U}, r, \boldsymbol{u}_r) \geq (1 - \eta_{j,\ell})^{2^{r-\ell}} d^{2^{r-1}} \prod_{i=1}^{r-1} |U_i|^{2^{r-2}} \prod_{i=r}^{k} |U_i|^{2^{r-1}}.$$

Hence, a sufficiently small choice of $\eta_{j,\ell} > 0$ yields the conclusion of Claim 3.10 due to Fact 3.9 applied with $N = |U_r|^{2^{r-1}}$ and $a = d^{2^{r-1}} \prod_{i=1}^{r-1} |U_i|^{2^{r-2}} \prod_{i=r+1}^{k} |U_i|^{2^{r-1}}$. $\square$

After those preparations we finally prove Lemma 3.8. Let k, d, and ε be given. We determine $\delta > 0$ as follows: Set $\gamma_{1,1} = \varepsilon/4$ and for $j = 2, \dots, k$ let

$$\gamma_{j,1} = \tfrac{1}{2}(d\varepsilon)^{2^{j-1}}\eta_{j-1,1}\,,$$

where $\eta_{j-1,1}$ is given by Claim 3.10 applied for $j-1$, $\ell = 1$ with $\gamma_{j-1,1}$. We then set $\delta = \eta_{k,1}/2$ and let n_0 be sufficiently large.

Suppose the k-uniform hypergraph H with vertex set V satisfies $\mathrm{MIN}_d(\delta)$. We have to show that H satisfies $\mathrm{DISC}_d(\varepsilon)$. For that fix an arbitrary set $U \subseteq V$. We have to show that

$$e(U) = d\binom{|U|}{k} \pm \varepsilon n^k\,. \tag{3.11}$$

This claim is trivial for sets U of size at most εn, so we assume $|U| \geq \varepsilon n$.

We are going to apply Claim 3.10 k times. We start with $j = k$, $\ell = 1$, and $\mathcal{U}_k = (U_{k,1}, \dots, U_{k,k})$, where all sets $U_{k,i}$ are equal to V for $i = 1, \dots, k$. Note that the property $\mathrm{MIN}_d(\delta)$ shows that for sufficiently large n the assumptions (a) and (b) of Claim 3.10 are satisfied by H. Recall, that $M_0 = K_k$ consists of one edge and

$$\hom(M_0, H, (V, \dots, V)) = k!e(H)$$

here. Now the conclusion of Claim 3.10 for $r = k$ shows that, due to the choice of $\gamma_{k,1}$ and $|U| \geq \varepsilon n$, the assumption (b) of Claim 3.10 for $j = k-1$, $\ell = 1$, and $\mathcal{U}_{k-1} = (U_{k-1,1}, \dots, U_{k-1,k})$ with $U_{k-1,i} = V$ for $i = 1, \dots, k-1$ and $U_{k-1,k} = U$ is met.

Moreover, noting that in general if $U_1 = U_i$, then

$$\hom(M_0, H, \mathcal{U}, 1, (u)) = \hom(M_0, H, \mathcal{U}, i, (u))$$

for every $u \in U_1 = U_i$, we see that conclusion of Claim 3.10 for $r = 1$ applied for $j = k$, $\ell = 1$, and $\mathcal{U}_k$, yields the assumption (a) of Claim 3.10 for $j = k-1$, $\ell = 1$, and $\mathcal{U}_{k-1}$.

In general we apply Claim 3.10 for $j = k, \dots, 1$, always with $\ell = 1$, and $\mathcal{U}_j = (U_{j,1}, \dots, U_{j,k})$, where $U_{j,1} = \dots = U_{j,j} = V$ and $U_{j,j+1} = \dots = U_{j,k} = U$ and observe, as above, that the conclusion of Claim 3.10 for j yield the assumptions for $j-1$.

This way the conclusion of the last application of Claim 3.10 for $j = \ell = 1$ and $r = 1$ gives a lower and an upper bound for $\hom(M_0, H, (V, U, \dots, U), 1, (u))$ for all but at most $\gamma_{1,1}|V|$ vertices of $u \in V$. Consequently,

$$\begin{aligned} k!e(U) &= \sum_{u \in U} \hom(M_0, H, (V, U, \dots, U), 1, (u)) \\ &\qquad = |U|(1 \pm \gamma_{1,1})d|U|^{k-1} \pm \gamma_{1,1}|V||U|^{k-1} = d|U|^k \pm \tfrac{\varepsilon}{2}n^k\,, \end{aligned}$$

which yields (3.11) for sufficiently large n. □

3.2.3 DEV_d implies DISC_d

In this section we verify another of the key implications of Theorem 1.3, by showing that DEV_d implies DISC_d.

Lemma 3.11. *For every integer $k \geq 2$, every $d > 0$, and every $\varepsilon > 0$, there exist $\delta > 0$ and n_0 such that the following is true. If H is a k-uniform hypergraph on $n \geq n_0$ vertices that satisfies* $\mathrm{DEV}_d(\delta)$*, then H satisfies* $\mathrm{DISC}_d(\varepsilon)$*.*

Proof. For given k, d and ε we set $\delta = (\varepsilon/4)^{2^k}$ and n_0 sufficiently large. Let H be a k-uniform hypergraph with vertex set $V = V(H)$ and $|V| = n \geq n_0$, which satisfies $\mathrm{DEV}_d(\delta)$. We want to verify $\mathrm{DISC}_d(\varepsilon)$ and for that let $U \subseteq V$ be a subset of vertices. Again we may assume without loss of generality that $|U| \geq \varepsilon n$.

Again, as in Section 3.2.2, we consider homomorphisms of M (and its subhypergraphs) instead of labeled copies. Additionally to the notation from Section 3.2.2, we denote by $\mathcal{V} = (V, \dots, V)$ the vector which contains the vertex set V k times. Moreover, we denote by K_V the complete k-uniform hypergraph with vertex set V. Recall that $w\colon E(K_V) \to [-1, 1]$, where $w(e) = 1 - d$ if $e \in E(H)$ and $w(e) = -d$ otherwise. We introduce $f(M_j, H, U)$, which is a short hand notation for the total weight of all homomorphisms of M_j into K_V with the property that the "last" $k - j$ vertex classes $X_{j+1}(M_j), \dots, X_k(M_j)$ of M_j are mapped into U. More precisely, for $j = 0, \dots, k$ we set

$$f(M_j, H, U) = \sum_{\varphi \in \mathrm{Hom}(M_j, K_V, \mathcal{V})} \prod_{e \in E(M_j)} w(\varphi(e)) \prod_{i=j+1}^{k} \prod_{x \in X_i(M_j)} \mathbf{1}_U(\varphi(x)), \tag{3.12}$$

where $\mathbf{1}_U$ denotes the indicator function of U. Fixing first the image of $X_{j+1}(M_j)$ and summing over all homomorphisms φ which extend this choice to a full homomorphism of M_j, we can rewrite $f(M_j, H, U)$ as follows

$$\sum_{\boldsymbol{v} \in V^{2^j}} \prod_{i=1}^{2^j} \mathbf{1}_U(v_i) \sum_{\varphi \in \mathrm{Hom}(M_j, K_V, \mathcal{V}, j+1, \boldsymbol{v})} \prod_{e \in E(M_j)} w(\varphi(e)) \prod_{i=j+2}^{k} \prod_{x \in X_i(M_j)} \mathbf{1}_U(\varphi(x)) .$$

Recalling, that $M_{j+1} = \mathrm{db}_{j+1}(M_j)$, i.e., M_{j+1} arises from M_j by fixing the $(j+1)$-st vertex class $X_{j+1}(M_j)$ of M_j and "doubling" all the edges together with the remaining vertices, and applying the Cauchy-Schwarz inequality to $f(M_j, H, U)$ (to the form stated above), we obtain

$$\left(f(M_j, H, U)\right)^2 \leq |U|^{2^j} f(M_{j+1}, H, U)$$

for every $j \in \{0, \dots, k-1\}$ and, consequently,

$$\left(f(M_j, H, U)\right)^{2^{k-j}} \leq |U|^{2^{k-1}} \left(f(M_{j+1}, H, U)\right)^{2^{k-j-1}} .$$

Applying the last inequality inductively for $j = 0, \dots, k-1$ we obtain

$$\left|f(M_0, H, U)\right|^{2^k} \leq |U|^{k 2^{k-1}} \left|f(M_k, H, U)\right| . \tag{3.13}$$

Since M_0 consists of a single edge we have

$$f(M_0, H, U) = k!e(U) - dk!\tbinom{|U|}{k} = k!e(U) - d|U|^k \pm \delta n^k\,,$$

since $|U| \geq \varepsilon n$ and n is sufficiently large. On the other hand, since $M_k = M$ we have for sufficiently large n

$$f(M_k, H, U) = \sum_{\varphi \in \mathrm{Hom}(M, K_V, \mathcal{V})} \prod_{e \in E(M)} w(\varphi(e)) = \sum_{\widetilde{M}} \prod_{e \in E(\widetilde{M})} w(\varphi(e)) \pm \delta n^{|V(M)|},$$

where the sum runs over all copies $\widetilde{M}$ of M in K_V. Since H satisfies $\mathrm{DEV}_d(\delta)$ we obtain for sufficiently large n

$$|f(M_k, H, U)| \leq 2\delta n^{|V(M)|}$$

and consequently (3.13) yields

$$|k!e(U) - d|U|^k| \leq (\delta + (2\delta)^{1/2^k})n^k$$

which implies

$$e_H(U) = d\tbinom{|U|}{k} \pm \varepsilon n^k,$$

for sufficiently large n by our choice of δ. □

3.2.4 Equivalence of MIN_d and MDEG_d

In this section we verify the equivalence of MIN_d and MDEG_d. As we will see the implication from MIN_d to MDEG_d is quite straightforward. Moreover, the reverse implication would be trivial, if MDEG_d would comprise the assumption that $e(H) \geq d\binom{n}{k} - o(n^k)$. In fact, in the main part of the proof we will deduce that k-uniform hypergraphs having MDEG_d must have the right density.

Lemma 3.12. *For every integer $k \geq 2$, every $d > 0$, and every ε, $\varepsilon' > 0$, there exists δ, $\delta' > 0$ and n_0 such that the following is true.*

(i) If H is a k-uniform hypergraph on $n \geq n_0$ vertices that satisfies $\mathrm{MIN}_d(\delta)$, then H satisfies $\mathrm{MDEG}_d(\varepsilon)$.

(ii) If H is a k-uniform hypergraph on $n \geq n_0$ vertices that satisfies $\mathrm{MDEG}_d(\delta')$, then H satisfies $\mathrm{MIN}_d(\varepsilon')$.

Proof. We start with the proof of (i). Let k, d and ε be given. We set $\gamma_{k,1} = \varepsilon/4$ and we let $\eta_{k,1}$ be given by Claim 3.10 applied with $j = k$ and $\gamma_{k,1}$. Then set $\delta = \eta_{k,1}/2$ and let n_0 be sufficiently large.

Let H be a k-uniform hypergraph on n vertices satisfying $\mathrm{MIN}_d(\delta)$, i.e., $e(H) \geq d\binom{n}{k} - \delta n^k$ and $N_M(H) \leq d^{e(M)} n^{|V(M)|} + \delta n^{|V(M)|}$ and, consequently, for sufficiently large n we have

$$\hom(M_0, H, \mathcal{V}) \geq dn^k - 2\delta n^k$$

and

$$\hom(M_k, H, \mathcal{V}) \leq d^{e(M_k)} n^{|V(M_k)|} + 2\delta n^{|V(M_k)|}\,.$$

Hence, the conclusion of Claim 3.10 implies that

$$\mathrm{ext}(M_{k-1}, H, \boldsymbol{u}) = \hom(M_{k-1}, H, \mathcal{V}, k, \boldsymbol{u}) \pm \tfrac{\varepsilon}{4} n^{(k-1)2^{k-2}} = (d^{2^{k-1}} \pm \tfrac{\varepsilon}{2}) n^{(k-1)2^{k-2}}$$

for all but at most $\gamma_{k,1} n^{2^{k-1}}$ labeled subsets $\boldsymbol{u}_k = (u_1, \dots, u_{2^{k-1}})$ of 2^{k-1} vertices in V. Therefore, from our choice of $\gamma_{k,1} \leq \varepsilon/4$ we obtain

$$\sum_{\boldsymbol{u}} \left| \mathrm{ext}(M_{k-1}, H, \boldsymbol{u}) - d^{2^{k-1}} n^{(k-1)2^{k-2}} \right| \leq \varepsilon n^{(k+1)2^{k-2}}\,,$$

where the sum runs over all labeled 2^{k-1}-element subsets $\boldsymbol{u}$ of V. This shows that H satisfies $\mathrm{MDEG}_d(\varepsilon)$ and concludes the proof of (i) from the lemma.

For the second implication of the lemma, we first note that, due to

$$N_M(H) \leq \sum_{\boldsymbol{u}} \left(\mathrm{ext}(M_{k-1}, H, \boldsymbol{u})\right)^2$$

property $\mathrm{MDEG}_d(\delta')$, for sufficiently small choice of δ', immediately implies

$$N_M(H) \leq d^{2^k} n^{k2^{k-1}} + \varepsilon' n^{k2^{k-1}}\,.$$

Consequently, we have to show that $\mathrm{MDEG}_d(\delta')$ also implies $e(H) \geq d\binom{n}{k} - \varepsilon' n^k$. For that we will verify the following claim.

Claim 3.13. *For all integers $k-1 \geq j \geq 1$, every $d > 0$ and every $\gamma_j > 0$, there exists $\eta_j > 0$ such that the following is true. If*

$$\sum_{\boldsymbol{u}_{j+1} \in V^{2^j}} \left| \hom(M_j, H, \mathcal{V}, j+1, \boldsymbol{u}_{j+1}) - d^{2^j} n^{|V(M_j)| - 2^j} \right| \leq \eta_j n^{|V(M_j)|} \tag{3.14}$$

for $\mathcal{V} = (V, \dots, V)$, then

$$\sum_{\boldsymbol{u}_j \in V^{2^{j-1}}} \left| \hom(M_{j-1}, H, \mathcal{V}, j, \boldsymbol{u}_j) - d^{2^{j-1}} n^{|V(M_{j-1})| - 2^{j-1}} \right| \leq \gamma_j n^{|V(M_{j-1})|}\,.$$

Before we verify Claim 3.13, we deduce part (ii) of Lemma 3.12 from the claim. For given $\varepsilon' > 0$ let $\gamma_1 = \varepsilon'/2$ and for $j = 1, \dots, k-1$ let η_j be given by Claim 3.13 applied with γ_j and set $\gamma_{j+1} = \eta_j$. Finally, set $\delta' = \eta_{k-1}/2$ and let n_0 be sufficiently large. From the assumption $\mathrm{MDEG}_d(\delta')$, standard calculations show that the assumption of Claim 3.13 for $j = k-1$ is satisfied and the conclusion yields the assumption for the claim with $j = k-2$. Repeating this argument for $j = k-2, \dots, 1$ we infer

$$\sum_{u \in V} \left| \hom(M_0, H, \mathcal{V}, 1, (v)) - dn^{k-1} \right| \leq \gamma_1 n^k = \frac{\varepsilon'}{2} n^k\,,$$

which yields $e(H) = d\binom{n}{k} \pm \varepsilon' n^k$ for sufficiently large n. $\square$

Proof of Claim 3.13. For given γ_j let η_j be sufficiently small, determined later. For $\boldsymbol{u}_j \in V^{2^{j-1}}$ set

$$\hom(M_{j+1}, H, \mathcal{V}, j+1, \boldsymbol{u}_j) = \sum_{\boldsymbol{u}'_j \in V^{2^{j-1}}} \hom(M_{j+1}, H, \mathcal{V}, j+1, (\boldsymbol{u}_j, \boldsymbol{u}'_j)),$$

i.e., $\hom(M_{j+1}, H, \mathcal{V}, j+1, \boldsymbol{u}_j)$ denotes the number of homomorphisms φ from M_{j+1} to H, where the "first" 2^{j-1} vertices of $X_{j+1}(M_{j+1})$ are mapped to $\boldsymbol{u}_j$. Here we have to clarify what we mean by "first" 2^{j-1} vertices. By that we mean those vertices in $X_{j+1}(M_{j+1})$ which form $X_{j+1}(M_{j-1})$, i.e., the originals before the j-th "doubling" step. First we observe

$$\hom(M_{j+1}, H, \mathcal{V}, j+1, \boldsymbol{u}_j) = \sum_{\boldsymbol{u}'_j \in V^{2^{j-1}}} \big(\hom(M_j, H, \mathcal{V}, j+1, (\boldsymbol{u}_j, \boldsymbol{u}'_j))\big)^2 \tag{3.15}$$

and the assumption of the claim enables us to control the right-hand side of (3.15). Indeed, due to the assumption of the claim we know that for all but at most $\sqrt[4]{\eta_j} n^{2^{j-1}}$ vectors $\boldsymbol{u}_j \in V^{2^{j-1}}$ there exist at most $\sqrt[4]{\eta_j} n^{2^{j-1}}$ vectors $\boldsymbol{u}'_j \in V^{2^{j-1}}$ such that

$$\big|\hom(M_j, H, \mathcal{V}, j+1, (\boldsymbol{u}_j, \boldsymbol{u}'_j)) - d^{2^j} n^{|V(M_j)|-2^j}\big| \geq \sqrt{\eta_j} n^{|V(M_j)|-2^j}$$

and we call such vectors $\boldsymbol{u}_j \in V^{2^{j-1}}$ *deviant.* For a non-deviant vector $\boldsymbol{u}_j \in V^{2^{j-1}}$ we infer from (3.15)

$$\begin{aligned}
&\hom(M_{j+1}, H, \mathcal{V}, j+1, \boldsymbol{u}_j) \\
&\qquad = n^{2^{j-1}} d^{2^{j+1}} n^{2|V(M_j)|-2^{j+1}} \pm (3\sqrt{\eta_j} + \sqrt[4]{\eta_j}) n^{2^{j-1}} n^{2|V(M_j)|-2^{j+1}} \\
&\qquad = (d^{2^{j+1}} \pm 4\sqrt[4]{\eta_j}) n^{2|V(M_j)|-2^{j+1}+2^{j-1}}\,.
\end{aligned} \tag{3.16}$$

On the other hand, for all $\boldsymbol{u}_j \in V^{2^{j-1}}$, we have

$$\hom(M_{j+1}, H, \mathcal{V}, j+1, \boldsymbol{u}_j) = \hom(M_{j+1}, H, \mathcal{V}, j, \boldsymbol{u}_j), \tag{3.17}$$

where $\hom(M_{j+1}, H, \mathcal{V}, j, \boldsymbol{u}_j)$ denotes the number of homomorphisms φ from M_{j+1} to H, where the "first" 2^{j-1} vertices of $X_j(M_{j+1})$ are mapped to $\boldsymbol{u}_j$. Again, by "first" 2^{j-1} vertices we mean those vertices in $X_j(M_{j+1})$ which form $X_j(M_{j-1}) = X_j(M_j)$, i.e., those vertices which are fixed in the j-th "doubling" step, see also Remark 3.2. Now, we further rewrite $\hom(M_{j+1}, H, \mathcal{V}, j, \boldsymbol{u}_j)$ and observe that it equals

$$\begin{aligned}
&\hom(M_{j+1}, H, \mathcal{V}, j, \boldsymbol{u}_j) \\
&\qquad = \sum_{(\varphi,\varphi')} \hom(M_j, H, \mathcal{V}, j+1, (\varphi(X_{j+1}(M_{j-1})), \varphi'(X_{j+1}(M_{j-1}))),
\end{aligned} \tag{3.18}$$

where the sum is indexed by all pairs of homomorphisms

$$(\varphi, \varphi') \in (\mathrm{Hom}(M_{j-1}, H, \mathcal{V}, j, \boldsymbol{u}_j))^2,$$

i.e., over all those pairs of homomorphisms each of which extends $\boldsymbol{u}_j$ to a homomorphic image of M_{j-1}. The identity simply says that we obtain all homomorphic images of M_{j+1} which extend $\boldsymbol{u}_j$ as the first 2^{j-1} vertices in $X_j(M_{j+1})$ by taking two homomorphic extensions of $\boldsymbol{u}_j$ to M_{j-1} (to obtain a homomorphic image of M_j) and attaching another homomorphic image of M_j to the image to the thereby fixed images of $X_{j+1}(M_j)$. From (3.16) we obtain another possibility to apply the assumption of the claim and more importantly to connect it with the conclusion. Note that, given the fixed choice of $\boldsymbol{u}_j$ and $X_{j+1}(M_j)$, there are at most $n^{|V(M_j)|-2^{j-1}-2^j}$ ways to attach such a copy of M_j. Therefore, the assumption combined with (3.18) yields

$$\begin{aligned} \mathrm{hom}(M_{j+1}, H, \mathcal{V}, j, \boldsymbol{u}_j) = \big(\mathrm{hom}(M_{j-1}, H, \mathcal{V}, j, \boldsymbol{u}_j)\big)^2 \times d^{2^j} n^{|V(M_j)|-2^j} \\ \pm n^{|V(M_j)|-2^{j-1}-2^j} \times \eta_j n^{|V(M_j)|}. \end{aligned} \tag{3.19}$$

Combining (3.16), (3.17), and (3.19), we obtain, for non-deviant vectors $\boldsymbol{u}_j \in V^{2^{j-1}}$,

$$\big(\mathrm{hom}(M_{j-1}, H, \mathcal{V}, j, \boldsymbol{u}_j)\big)^2 = (d^{2^j} \pm (4\sqrt[4]{\eta_j} + \eta_j)/d^{2^j}) n^{|V(M_j)|-2^{j-1}}$$

and, consequently, for sufficiently small choice of η_j (compared to γ_j and d) we have

$$\Big|\mathrm{hom}(M_{j-1}, H, \mathcal{V}, j, \boldsymbol{u}_j) - d^{2^{j-1}} n^{|V(M_{j-1})|-2^{j-1}}\Big| \le \frac{\gamma_j}{2} n^{|V(M_{j-1})|-2^{j-1}}$$

for non-deviant $\boldsymbol{u}_j \in V^{2^{j-1}}$. Summing over all $\boldsymbol{u}_j \in V^{2^{j-1}}$ we get

$$\begin{aligned} \sum_{\boldsymbol{u}_j \in V^{2^{j-1}}} \Big|\mathrm{hom}(M_{j-1}, H, \mathcal{V}, j, \boldsymbol{u}_j) - d^{2^{j-1}} n^{|V(M_{j-1})|-2^{j-1}}\Big| \\ \le \frac{\gamma_j}{2} n^{|V(M_{j-1})|} + \sqrt[4]{\eta_j} n^{|V(M_{j-1})|} \le \gamma_j n^{|V(M_{j-1})|} \end{aligned}$$

as claimed.

□

3.2.5 DEV_d implies CL_d

In this section we give a direct proof of $\mathrm{DEV}_d \Longrightarrow \mathrm{CL}_d$. For that we will introduce another version of DISC_d called FDISC_d, which is motivated by the quasi-random functions introduced by Gowers in [Gow06, see Section 3]. We show the following implications: $\mathrm{DEV}_d \Longrightarrow \mathrm{FDISC}_d$ (Lemma 3.14), $\mathrm{DISC}_d \Longrightarrow \mathrm{FDISC}_d$ (Lemma 3.15) and $\mathrm{FDISC}_d \Longrightarrow \mathrm{CL}_d$ (Lemma 3.16).

Before we define FDISC_d, we will generalize the weight function w defined in Section 3.1. For a k-uniform hypergraph H with vertex set V and some $d \in [0, 1]$, we define the weight function $w\colon [V]^{\leq k} \to [-1,1]$ as follows: for a set $X \subseteq V$ of cardinality at most k we set

$$w(X) = \begin{cases} 1-d & \text{if } X \in E(H), \\ -d & \text{otherwise.} \end{cases}$$

Our weight function is now applicable also to subsets of cardinality smaller than k. This generalization will simplify the notation. Moreover, we will again use homomorphisms instead of copies of k-uniform hypergraphs. In this section we study the following properties.

$\mathrm{FDISC}_d(\varepsilon)$ We say a k-uniform hypergraph H on n vertices has $\mathrm{FDISC}_d(\varepsilon)$ for d, $\varepsilon > 0$, if

$$\left| \sum_{\varphi\colon [k]\to V(H)} w(\varphi([k])) \prod_{i=1}^{k} g_i(\varphi(i)) \right| \leq \varepsilon n^k$$

for all families of functions $g_i\colon V(H) \to [-1,1]$ with $i \in [k]$.

For convenience we will work with the following version of DEV_d.

$\mathrm{DEV}'_d(\varepsilon)$ We say a k-uniform hypergraph H_n on n vertices has $\mathrm{DEV}'_d(\varepsilon)$ for d, $\varepsilon > 0$, if

$$\left| \sum_{\varphi\colon V(M)\to V} \prod_{e\in E(M)} w(\varphi(e)) \right| \leq \varepsilon n^{k2^{k-1}}.$$

This definition, though formally different to the definition of DEV_d, is equivalent to it. For DEV_d we were summing over all labeled copies of M in K_V, and here we sum over all mappings from $V(M)$ to V (note that we extended w to $[V]^{\leq k}$ for that). By doing this, we get at most an additional additive error term of $O(n^{k2^{k-1}-1}) = o(n^{k2^{k-1}})$, which is asymptotically negligible.

Lemma 3.14. *For every integer $k \geq 2$, every $d > 0$, and every $\varepsilon > 0$ there exist $\delta > 0$ and n_0 such that the following is true. If H is a k-uniform hypergraph on $n \geq n_0$ vertices that satisfies* $\mathrm{DEV}'_d(\delta)$*, then H satisfies* $\mathrm{FDISC}_d(\varepsilon)$.

Proof. The assertion $\mathrm{DEV}_d \Rightarrow \mathrm{FDISC}_d$ is a simple generalization of the proof of our Lemma 3.11. We only have to replace $\mathbf{1}_U(\varphi(x))$ for $x \in X_i(M_j)$ by $g_i(\varphi(x))$. Thus, applying each time the Cauchy-Schwarz inequality we will square $g_i(\varphi(x))$, and we then only have to upper bound $(g_i(\varphi(x)))^2$ by 1. We also now have to sum over all functions $\varphi\colon V(M_j) \to V$ (instead of over all homomorphisms $\varphi \in \mathrm{Hom}(M_j, K_V, \mathcal{V})$). With those adjustments the proof works verbatim. □

The property FDISC_d is easily seen to imply DISC_d, all one has to do is to choose all functions $g_i = \mathbf{1}_U$ for any subset $U \subseteq V$. Here we give a slightly sketchy proof of the reverse implication as well.

Lemma 3.15. *For every integer $k \geq 2$, every $d > 0$, and every $\varepsilon > 0$ there exist $\delta > 0$ and n_0 such that the following is true. If H is a k-uniform hypergraph on $n \geq n_0$ vertices that satisfies* $\mathrm{DISC}_d(\delta)$*, then H satisfies* $\mathrm{FDISC}_d(\varepsilon)$.

Proof (sketch). First we note that FDISC_d is equivalent to the property when we allow functions g_i, $i \in [k]$, to take values in $[0,1]$ (and this can be seen by setting every $g_i = g_i^+ - g_i^-$ with $g_i^+, g_i^- : V \to [0,1]$ and then applying the triangle inequality). We call this property FDISC'_d and let us assume that DISC does not imply FDISC'_d. Therefore there exist functions $g_i \colon V(H) \to [0,1]$ such that

$$\left| \sum_{x_i \in V(H), i \in [k]} w(\{x_1, \ldots, x_k\}) \prod_{i=1}^{k} g_i(x_i) \right| \geq \varepsilon n^k.$$

Now we let $X_i, i \in [k]$, be random subsets, whose elements are chosen independently with probabilities given by the functions $g_i, i \in [k]$. Then the left hand side of the above inequality is the expectation in absolute value of

$$\sum_{x_i \in X_i, i \in [k]} w(\{x_1, \ldots, x_k\})$$

Therefore we deduce that there exist subsets $X'_1, \ldots, X'_k$ of $V(H)$ such that

$$\left| \sum_{x_i \in X'_i, i \in [k]} w(\{x_1, \ldots, x_k\}) \right| \geq \varepsilon n^k.$$

But we can also rewrite the above sum (by noting that $w(e) = \mathbf{1}_{E(H)}(e) - d$ for any $e \in [V(H)]^{\leq k}$) as

$$\left| \hom(K_k^{(k)}, H, (X'_1, \ldots, X'_k)) - d \prod_{i \in [k]} |X'_i| \right| \geq \varepsilon n^k.$$

Now, if for all X'_i, X'_j's it holds that they are either equal or disjoint, then we immediately obtain a contradiction to $\mathrm{DISC}_{d,\tau}(\varepsilon/2)$ for some appropriate type $\tau \colon [k] \to [\ell]$, where $\ell \in [k]$ (see Theorem 3.4). If this is not the case, one can easily show by induction on the number

$$\left| \{(X'_i, X'_j) \colon X'_i \neq X'_j \text{ and } X'_i \cap X'_j \neq \varnothing\} \right|,$$

that this implies the existence of some other sets $Y_1, \ldots, Y_k$, either disjoint or equal, such that some $\mathrm{DISC}_{d,\tau}(\varepsilon')$ is violated for some positive ε', which will constitute a contradiction to Theorem 3.4 (see also Section 3.4). □

We close this section with the proof of the implication $\mathrm{FDISC}_d \Rightarrow \mathrm{CL}_d$.

Lemma 3.16. *For every integer $k \geq 2$, every $d > 0$, every linear k-uniform hypergraph F on ℓ vertices, and every $\varepsilon > 0$, there exist $\delta > 0$ and n_0 such that the following is true. If H is a k-uniform hypergraph on $n \geq n_0$ vertices that satisfies* $\mathrm{FDISC}_d(\delta)$*, then H satisfies* $\mathrm{CL}_d(F, \varepsilon)$.

Proof. We may assume $E(F) \neq \varnothing$. It suffices to verify an estimate on

$$\hom(F,H) = \sum_{\varphi \in \mathrm{Hom}(F,K_V,\mathcal{V})} \prod_{e \in E(F)} \mathbf{1}_{E(H)}(\varphi(e)). \tag{3.20}$$

the number of homomorphisms from F into H, where $\mathcal{V} = (V,\dots,V)$. Here, again, we may further enlarge the sum by going over all functions $\varphi\colon V(F) \to V$. However, for every φ which is not a homomorphism, there will be an $f \in E(F)$ with $|\varphi(f)| < k$, and thus φ will contribute 0 to the total sum. Noting furthermore that $\mathbf{1}_{E(H)}(\varphi(e)) = w(\varphi(e)) + d$ for every $\varphi(e) \in [V]^{\leq k}$ we may rewrite (3.20) as

$$\hom(F,H) = \sum_{\varphi\colon\, V(F)\to V} \prod_{e \in E(F)} (w(\varphi(e)) + d)$$

Multiplying out the inner product $\prod_{e\in E(F)}(w(\varphi(e))+d)$, we obtain the main term, which is $d^{e(F)}n^{\ell}$, while each of the remaining $2^{e(F)} - 1$ terms is of the form

$$\prod_{e \in E(F)} q_e,$$

where every q_e is either d or $w(\varphi(e))$, and at least one $q_f = w(\varphi(f))$ for some $f \in E(F)$. We can write each such term as

$$\sum_{\varphi\colon\, V(F)\to V} \prod_{e\in E(F)} q_e = \sum_{\varphi'\colon V(F)\setminus\{f\}\to V} \; \sum_{\substack{\varphi\colon V(F)\to V\\ \varphi|_{V(F)\setminus\{f\}}=\varphi'}} w(\varphi(f)) \prod_{e\in E(F)\setminus\{f\}} q_e.$$

Now we interpret the product on the right hand side as a product of functions g_i (for every vertex i of f since F is linear) and we can apply $\mathrm{FDISC}_d(\delta)$ to obtain an estimate for the inner sum. Therefore, setting $\delta = \varepsilon/2^{e(F)+1}$, we have shown that the inner sum is $d^{e(F)}n^k \pm \varepsilon n^k/2$ and, hence,

$$\hom(F,H) = d^{e(F)}n^{\ell} \pm \varepsilon n^{\ell}\,,$$

which implies $\mathrm{CL}_d(F,\varepsilon)$ for sufficiently large n. □

3.3 Proof of Theorem 3.3

In this section we present the proof of Theorem 3.3. We have to show that for every $k \geq 2$, every linear k-uniform hypergraph F with at least one edge and $V(F) = [\ell]$ for some integer ℓ, every $d > 0$, and every vector $\boldsymbol{\alpha} \in (0,1)^{\ell}$ with $\sum_i \alpha_i < 1$ the properties DISC_d, $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}$ and $\mathrm{HCL}_{d,F}$ are equivalent. In Section 3.3.1 we show the simple implication $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}} \Longrightarrow \mathrm{HCL}_{d,F}$ (Fact 3.17). The main part of this section is devoted to the proof of $\mathrm{HCL}_{d,F} \Longrightarrow \mathrm{DISC}_d$. For that we will introduce another property REG_d, which will turn out to be equivalent to DISC_d and we then show $\mathrm{HCL}_{d,F} \Longrightarrow \mathrm{REG}_d$ in Section 3.3.2, that $\mathrm{HCL}_{d,F}$ implies REG_d (Lemma 3.22) and REG_d is equivalent to DISC_d (Fact 3.21).

Finally, in Section 3.3.3 we verify that DISC_d implies $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}$ (Fact 3.24).

3.3.1 $\mathrm{HCL}_{d,F,\alpha}$ implies $\mathrm{HCL}_{d,F}$

The following observation yields the implication from $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}$ to $\mathrm{HCL}_{d,F}$.

Fact 3.17. *For every integer $k \geq 2$, every $d > 0$, every linear k-uniform hypergraph F with at least one edge and $V(F) = [\ell]$ for some integer ℓ, all vectors $\boldsymbol{\alpha} \in (0,1)^\ell$ with $\sum_{i=1}^{\ell} \alpha_i < 1$, and every $\varepsilon > 0$, there exist $\delta > 0$ and n_0 such that the following is true. If H is a k-uniform hypergraph on $n \geq n_0$ vertices that satisfies $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}(\delta)$, then, for all $\boldsymbol{\beta} \in (0,1)^\ell$ with $\sum_{i=1}^{\ell} \beta_i < 1$, H satisfies $\mathrm{HCL}_{d,F,\boldsymbol{\beta}}(\varepsilon)$.*

Proof. Note that it suffices to consider the case when $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_\ell)$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_\ell)$ differ in at most one entry, i.e., there is an $i \in [\ell]$ such that $\alpha_i \neq \beta_i$ and for all $j \neq i$ we have $\alpha_j = \beta_j$. Without loss of generality we may assume that $i = \ell$. For given k, d, F, $\boldsymbol{\alpha}$, and $\varepsilon > 0$ we set $\delta = \varepsilon \cdot \alpha_\ell \cdot (1 - \sum_i \alpha_i)/6$ and let n_0 be sufficiently large. We then verify the fact for given $\boldsymbol{\beta} \in (0,1)^\ell$.

First, we prove the claim for all $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{\ell-1}, \gamma)$ with $\gamma \geq \alpha_\ell$. Let $U_1, \dots, U_\ell \subseteq V(H)$ be subsets satisfying $|U_i| = \lfloor \beta_i n \rfloor$ for $i \in [\ell-1]$, $|U_\ell| = \lfloor \gamma n \rfloor$ and $\mathcal{P} = \{W \subset U_\ell \colon |W| = \lfloor \alpha_\ell n \rfloor\}$. Since H satisfies $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}(\delta)$ and $\beta_j = \alpha_j$ for all $j \in [\ell-1]$ we infer

$$N_F(U_1, \dots, U_{\ell-1}, W) = d^{e(F)} \lfloor \alpha_\ell n \rfloor \prod_{i \in [\ell-1]} |U_i| \pm \delta n^\ell$$

for all $W \in \mathcal{P}$. Hence, having each copy of F counted $\binom{\lfloor \gamma n \rfloor - 1}{\lfloor \alpha_\ell n \rfloor - 1}$ times, we obtain, for $n \geq 1/\alpha_\ell$,

$$\begin{aligned}
N_F(U_1, \dots, U_\ell) &= \binom{\lfloor \gamma n \rfloor - 1}{\lfloor \alpha_\ell n \rfloor - 1}^{-1} \sum_{W \in \mathcal{P}} N_F(U_1, \dots, U_{\ell-1}, W) \\
&= \binom{\lfloor \gamma n \rfloor - 1}{\lfloor \alpha_\ell n \rfloor - 1}^{-1} \binom{\lfloor \gamma n \rfloor}{\lfloor \alpha_\ell n \rfloor} d^{e(F)} \left(\lfloor \alpha_\ell n \rfloor \prod_{i \in [\ell]} |U_i| \pm \delta n^\ell \right) \\
&= d^{e(F)} \prod_{i \in [\ell]} |U_i| \pm \frac{2\delta}{\alpha_\ell} n^\ell ,
\end{aligned}$$

which by our choice of δ yields the fact for this case.

Suppose $\beta_\ell < \alpha_\ell$. Without loss of generality we may assume that $\sum_{i \in [\ell]} \beta_i + \alpha_\ell < 1$. (Otherwise, first choose $\beta'_\ell = (1 - \sum_{i \in [\ell]} \alpha_i)/2$ and then use the proof from above to finish the claim for β_ℓ.) Let $U_1, \dots, U_\ell \subseteq V(H)$ be pairwise disjoint with $|U_i| = \lfloor \beta_i n \rfloor$, $i \in [\ell]$. Considering $W \subseteq V \setminus U_\ell$ of size $|W| = \lfloor \alpha_\ell n \rfloor$ we infer from $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}(\delta)$ and the case considered above

$$N_F(U_1, \dots, U_{\ell-1}, U_\ell \dot\cup W) = d^{e(F)} (\lfloor \alpha_\ell n \rfloor + \lfloor \beta_\ell n \rfloor) \prod_{i \in [\ell-1]} |U_i| \pm \frac{2\delta}{\alpha_\ell} n^\ell$$

and

$$N_F(U_1,\dots,U_{\ell-1},W) = d^{e(F)}\lfloor \alpha_\ell n\rfloor \prod_{i\in[\ell-1]} |U_i| \pm \delta n^\ell.$$

Hence, we have

$$\begin{aligned} N_F(U_1,\dots,U_\ell) &= N_F(U_1,\dots,U_{\ell-1},U_\ell \dot\cup W) - N_F(U_1,\dots,U_{\ell-1},W) \\ &= d^{e(F)} \prod_{i\in[\ell]} |U_i| \pm \frac{3\delta}{\alpha_\ell} n^\ell\,, \end{aligned}$$

which concludes the proof of the fact by the choice of δ. □

3.3.2 $\mathrm{HCL}_{d,F}$ implies DISC_d

In this section we verify the implication from $\mathrm{HCL}_{d,F}$ to DISC_d. The proof is based on ideas of Shapira and Yuster [SY08], the main tools being the theorem of Gottlieb [Got66] on the rank of the inclusion matrices and the weak regularity lemma for hypergraphs.

First we introduce the result of Gottlieb and its consequences. Then we consider another quasi-random property REG_d, which is equivalent to DISC_d. Finally, we prove that $\mathrm{HCL}_{d,F}$ implies REG_d.

Tools from linear algebra

For positive integers $r \ge \ell \ge k$ the inclusion matrix $I(r,\ell,k)$ is an $\binom{r}{\ell} \times \binom{r}{k}$ matrix defined as follows. For $L \in [r]^\ell$ and $K \in [r]^k$ the entry of $I_{L,K}$ is given by

$$I_{L,K} = \begin{cases} 1 & \text{if } K \subset L \\ 0 & \text{otherwise} \end{cases}$$

Note that we implicitly assume fixed orderings on the set of subgraphs $[r]^\ell$ and on the edge set $[r]^k$. This does not effect the rank of $I(r,\ell,k)$ which is at most $\binom{r}{k}$ and in fact it was shown by Gottlieb [Got66], that $I(r,\ell,k)$ has full rank if $r \ge \ell + k$.

Theorem 3.18 (Gottlieb)**.** *For all positive integers $\ell \ge k$ and $r \ge \ell + k$ the inclusion matrix $I(r,\ell,k)$ has rank $\binom{r}{k}$.* □

Note that the rows of $I(r,\ell,k)$ can be interpreted as incidence vectors of the edges of copies of the complete k-uniform hypergraph K_ℓ in K_r. For our purposes, it will be convenient to consider a similar matrix, where the rows correspond to incidence vectors of the edges of the given k-uniform hypergraph F. To this end, for a k-uniform hypergraph F on ℓ vertices, we define the matrix $A(r,F,k)$ as follows. The rows of $A(r,F,k)$ are indexed by the labeled copies of F in K_r and the columns are indexed, as above, by the k-element subsets of $[r]$. Now for a labeled copy $\widetilde{F}$ of F in K_r and a k-set $e \in [r]^k$ the entry $A_{\widetilde{F},e}$ is given by

$$A_{\widetilde{F},e} = \begin{cases} 1 & \text{if } e \in E(\widetilde{F}) \\ 0 & \text{otherwise.} \end{cases}$$

Thus $A(r, F, k)$ is a $N_F(K_r) \times [r]^k$ and Theorem 3.18 determines the rank of $A(r, F, k)$.

Corollary 3.19. *For all positive integers $\ell \geq k$, $r \geq \ell + k$ and all non-empty k-uniform hypergraphs F on ℓ vertices the matrix $A(r, F, k)$ has rank $\binom{r}{k}$.*

Proof. The proof of Corollary 3.19 is identical to the proof of Lemma 3.1 in [Sha10] and follows from the observation that the rows of $A(r, F, k)$ span the rows of $I(r, \ell, k)$. Indeed, summing all rows of $A(r, F, k)$ that correspond to copies $\widetilde{F}$ of F with the same vertex set $L \in [r]^\ell$ we obtain a multiple of the row in $I(r, \ell, k)$ indexed by L. $\square$

From Corollary 3.19 we deduce the key lemma of this section, Lemma 3.20 below. In Lemma 3.20 we consider complete, weighted k-uniform hypergraphs on r vertices. Let $w\colon E(K_r) \to (0, 1]$ be an arbitrary weight function and F be a fixed k-uniform hypergraph on ℓ vertices. We set the weight of a labeled copy $\widetilde{F}$ of F in K_r, as before, to be the product of the weights of the edges of $\widetilde{F}$, i.e.,

$$w(\widetilde{F}) = \prod_{e \in E(\widetilde{F})} w(e)\,.$$

Lemma 3.20 states that if $w(\widetilde{F})$ is "almost" the same for all copies of F, then w must be almost constant.

Lemma 3.20. *For all integers $\ell \geq k \geq 2$ and $r \geq \ell + k$, every $d > 0$, every k-uniform hypergraph F on ℓ vertices with at least one edge, and every $\varepsilon > 0$, there exists $\delta > 0$ such that if $w\colon E(K_r) \to (0, 1]$ satisfies*

$$w(\widetilde{F}) = d^{e(F)} \pm \delta$$

for all labeled copies $\widetilde{F}$ of F in K_r, then $w(e) = d \pm \varepsilon$ for all $e \in E(K_r)$.

Proof. Let ℓ, k, r, d, F, and ε be given. Due to the continuity of the function 2^x we can choose $\varepsilon' > 0$ such that if $|x - \log_2 d| \leq \varepsilon'$ then $|2^x - d| \leq \varepsilon$. Next we fix an ordering $e_1, \ldots, e_m$, $m = \binom{r}{k}$ of the edges of the K_r and an ordering $\widetilde{F}_1, \ldots, \widetilde{F}_t$ for $t = r(r-1)\ldots(r-\ell+1)$ of all labeled copies of F in K_r. This defines the matrix $A = A(r, F, k)$ which, by Corollary 3.19, has rank $\binom{r}{k}$. Thus $A\colon \mathbb{R}^{\binom{r}{k}} \to \mathbb{R}^t$ is an injective and linear function and consequently there exists a $\delta' > 0$ such that the following holds: if $A\boldsymbol{y} = \boldsymbol{b}$ and $A\boldsymbol{x} = \boldsymbol{c}$ with $\|\boldsymbol{b} - \boldsymbol{c}\|_\infty \leq \delta'$ then $\|\boldsymbol{y} - \boldsymbol{x}\|_\infty \leq \varepsilon'$. Further, due to the continuity of the function $\log_2 x$ we can choose $\delta > 0$ such that if $|2^b - d^{e(F)}| \leq \delta$, then $|b - e(F)\log_2 d| \leq \delta'$ and we fix the δ for Lemma 3.20 this way.

Now let $w\colon E(K_r) \to (0, 1]$ satisfy the assumption of the lemma. Therefore, we have for every copy $\widetilde{F}$ of F in K_r

$$\sum_{e \in E(\widetilde{F})} \log_2 w(e) = \log_2(d^{e(F)} \pm \delta)\,. \tag{3.21}$$

Let $\boldsymbol{y} = ((y(e_1), \ldots, y(e_m)) \in \mathbb{R}^m$ be given by

$$y(e_i) = \log_2 w(e_i)$$

for $i = 1, \dots, m$. Then (3.21) is equivalent to $A\boldsymbol{y} = \boldsymbol{b}$ where $\boldsymbol{b} = (b_1, \dots, b_t)$ with $b_i = \log_2(d^{e(F)} \pm \delta)$ for all $i \in [t]$.

On the other hand, by Corollary 3.19 we know that A has rank $\binom{r}{k}$ and, hence, the system of linear equations $A\boldsymbol{x} = \boldsymbol{c}$ for $\boldsymbol{c} = (e(F)\log d)\mathbf{1}_t$ for the all ones vector $\mathbf{1}_t = \{1\}^t$ has at most one solution. Since the everywhere $\log d$ vector $(\log_2 d)\mathbf{1}_m$ is a solution to this system of equations, it must be the unique solution $\boldsymbol{x}$.

From our choice of δ we infer $\|\boldsymbol{b} - \boldsymbol{c}\|_\infty \le \delta'$ and, consequently, due to the choice of δ' we have $\|\boldsymbol{y} - \boldsymbol{x}\|_\infty \le \varepsilon'$. In other words, $|\log_2(w(e_i)) - \log_2(d)| \le \varepsilon'$ for every $i = 1, \dots, m$ and the choice of ε' yields $|w(e) - d| \le \varepsilon$ for all edges $e \in E(K_r)$. $\square$

Property REG_d

For the proof of $\mathrm{HCL}_{d,F} \Rightarrow \mathrm{DISC}_d$ we will use the weak regularity lemma for k-uniform hypergraphs, Theorem 2.7. Roughly speaking, the property $\mathrm{HCL}_{d,F}$ will imply that for the weighted cluster-hypergraph of a regular partition the assumptions of Lemma 3.20 hold. Consequently, the densities of all k-tuples of the regular partition will be close to d and from this we will infer DISC_d. Below we briefly discuss the connection of REG_d and DISC_d.

In case of graphs, it was noted by Simonovits and Sós [SS91] that there is a close relationship between quasi-randomness and the Szemerédi regular partition. Indeed, it is easily shown that a graph G is quasi-random in the sense of Theorem 1.1 if and only if G permits a partition such that almost all pairs of partition classes are regular and have roughly the same density. This generalizes to k-uniform hypergraphs in a straightforward manner.

It will be convenient to consider the property REG_d defined as follows.

$\mathrm{REG}_d(\varepsilon)$ We say a k-uniform hypergraph H on n vertices has $\mathrm{REG}_d(\varepsilon)$ for d, $\varepsilon > 0$, if there exists an ε-regular, t-equipartition $V(H) = V_1 \dot\cup \dots \dot\cup V_t$ of H with $g(d,\varepsilon) \ge t \ge 1/\varepsilon$ for some arbitrary function $g(d,\varepsilon) \ge 1/\varepsilon$ independent of H and n such that $d(V_{i_1}, \dots, V_{i_k}) = d \pm \varepsilon$ for all but at most εt^k tuples $\{i_1, \dots, i_k\} \in \binom{[t]}{k}$.

It is easy to see that DISC_d and REG_d are equivalent (see, e.g. [Chu91]) and we omit the proof here.

Fact 3.21. *For every integer $k \ge 2$ and every $d > 0$ the properties DISC_d and REG_d are equivalent.* $\square$

$\mathrm{HCL}_{d,F}$ implies REG_d

In this section we deduce REG_d from $\mathrm{HCL}_{d,F}$ by proving the following lemma.

Lemma 3.22. *For every integer $k \ge 2$, every $d > 0$, every linear k-uniform hypergraph F containing at least one edge, and every $\varepsilon > 0$, there exists $\delta > 0$ and n_0 such that the following is true. If H is a k-uniform hypergraph on $n \ge n_0$ vertices that satisfies $\mathrm{HCL}_{d,F}(\delta)$, then H satisfies $\mathrm{REG}_d(\varepsilon)$.*

Besides the results from the sections above, we will also need the following consequence of a packing result of Rödl [Röd85].

Lemma 3.23. *For all integers $r \geq k \geq 2$ and every $\gamma > 0$ there exists an integer t_0 such that for all $t \geq t_0$ the following holds. If R is a k-uniform hypergraph on t vertices with $e(R) \geq (1-\gamma)\binom{t}{k}$ edges, then there exist at least $(1-\gamma r^k)\binom{t}{k}$ edges in R each of which belong to at least one copy of K_r in R.*

Proof. We choose t_0 large enough to guarantee that the packing result of Rödl [Röd85] is applicable for $t \geq t_0$ and r, k, and γ. Given a k-uniform hypergraph R on t vertices which contains at least $(1-\gamma)\binom{t}{k}$ edges we first consider the complete k-uniform hypergraph K_t on the same vertex set. From Rödl's theorem we infer that K_t contains at least $(1-\gamma)\binom{t}{k}/\binom{r}{k}$ edge disjoint copies of the K_r. Taking the same copies of K_r we see that at most $\gamma\binom{t}{k} = \gamma\binom{r}{k}\binom{t}{k}/\binom{r}{k}$ of them fail to be a subgraph of R since R contains at least $(1-\gamma)\binom{t}{k}$ edges. This implies that R contains at least $(1-\gamma-\gamma\binom{r}{k})\binom{t}{k}/\binom{r}{k}$ edge disjoint copies of K_r which implies that all but at most $\gamma r^k\binom{t}{k}$ edges of R are contained in a copy of a K_r in R. □

Proof of Lemma 3.22. For given k, d, linear k-uniform hypergraph F with at least one edge and $V(F) = [\ell]$, and $\varepsilon > 0$, we first apply Lemma 3.20 with ℓ, k, and $r = \ell + k$, d, F, and ε and obtain $\delta_{\mathrm{GL}} > 0$. Then we apply the counting lemma, Lemma 2.6, with ℓ, k, and $\gamma_{\mathrm{CL}} = \delta_{\mathrm{GL}}/2$ to obtain $\varepsilon_{\mathrm{CL}}$ and m_{CL}. Further, we apply Lemma 3.23 with r, k and $\gamma_{\mathrm{PL}} = \varepsilon/(2r^k)$ to obtain t_{PL}. Applying the weak regularity lemma, Theorem 2.7, with

$$\varepsilon_{\mathrm{RL}} = \min\{\varepsilon_{\mathrm{CL}}, \varepsilon/(2r^k)\} \quad \text{and} \quad t_0 = \max\{1/\varepsilon_{\mathrm{RL}}, t_{\mathrm{PL}}\}$$

we obtain T_0. Finally, we choose $\delta = \delta_{\mathrm{GL}} d^{e(F)}/(2^{\ell+2}T_0^{\ell})$ and $n_0 \geq T_0 m_{\mathrm{CL}}$ sufficiently large to satisfy the equations needed.

Let H be a k-uniform hypergraph on n vertices with $n \geq n_0$ which satisfies $\mathrm{HCL}_{d,F}(\delta)$. We have to show that there exists a partition $V_1 \dot\cup \ldots \dot\cup V_t = V(H)$ such that

(*i*) $1/\varepsilon \leq t \leq T_0$ (note that $T_0 = T_0(d, \varepsilon, F)$ is independent of H and n),

(*ii*) $||V_i| - |V_j|| \leq 1$ for all $i, j \in [t]$

(*iii*) all but at most εt^k k-tuples $(V_{i_1}, \ldots V_{i_k})$ are ε-regular and have density $d \pm \varepsilon$.

To this end, we first apply Theorem 2.7 with $\varepsilon_{\mathrm{RL}}$ and t_0 to obtain a partition $V(H) = V_1 \dot\cup \ldots \dot\cup V_t$, which already satisfies (*i*) and (*ii*) and the first part of (*iii*), i.e., all but at most $\varepsilon_{\mathrm{RL}}\binom{t}{k} \leq \frac{1}{2}\varepsilon t^k$ k-tuples $(V_{i_1}, \ldots V_{i_k})$ are ε-regular. Thus, it remains to show that all but at most $\frac{1}{2}\varepsilon t^k$ of the k-tuples $(V_{i_1}, \ldots V_{i_k})$ have density $d \pm \varepsilon$.

We consider the cluster hypergraph R, i.e., the k-uniform hypergraph on the vertex set $\{1, \ldots, t\}$ with $\{i_1, \ldots, i_k\}$ being an edge if and only if $(V_{i_1}, \ldots, V_{i_k})$ is $\varepsilon_{\mathrm{RL}}$-regular. Then R is a k-uniform hypergraph on t vertices which contains at least $(1-\varepsilon_{\mathrm{RL}})\binom{t}{k}$ edges and we assign to each edge $\{i_1, \ldots, i_k\}$ the weight

$$w(i_1, \ldots, i_k) = d(V_{i_1}, \ldots, V_{i_k})\,.$$

Applying Lemma 3.23 to R we know that all but at most $\gamma_{\mathrm{PL}} r^k \binom{t}{k} < \frac{1}{2}\varepsilon t^k$ edges belong to a copy of K_r in R. Thus, it is sufficient to show that every edge contained in a copy of K_r has weight $d \pm \varepsilon$.

For that fix a copy of K_r in R and without loss of generality we may assume that $V_1, \ldots, V_r$ are the vertices of that copy. Recall that H satisfies $\mathrm{HCL}_{d,F}(\delta)$ and as a consequence we have for every injective map $\varphi\colon [\ell] \to [r]$

$$N_F(V_{\varphi(1)}, \ldots, V_{\varphi(\ell)}) = d^{e(F)} \prod_{i\in[\ell]} |V_{\varphi(i)}| \pm \delta n^\ell \,.$$

Since each set $V_{\varphi(j)}$ has size at least $n/(2T_0)$ and $\delta = \delta_{\mathrm{GL}}/(2^{\ell+2}T_0^\ell)$, we obtain

$$N_F(V_{\varphi(1)}, \ldots, V_{\varphi(\ell)}) = \Big(d^{e(F)} \pm \delta_{\mathrm{GL}}/2\Big) \prod_{i\in[\ell]} |V_{\varphi(i)}| \,. \tag{3.22}$$

On the other hand, applying the counting lemma, Lemma 2.6, we obtain

$$N_F(V_{\varphi(1)}, \ldots, V_{\varphi(\ell)}) = \left(\prod_{e\in E(F)} w(\varphi(e)) \pm \gamma_{\mathrm{CL}}\right) \prod_{i\in[\ell]} |V_{\varphi(i)}| \,. \tag{3.23}$$

Combining (3.22) and (3.23) with the choice of $\gamma_{\mathrm{CL}} = \delta_{\mathrm{GL}}/2$ we conclude that

$$\prod_{e\in E(F)} w(\varphi(e)) = d^{e_F} \pm \delta_{\mathrm{GL}}$$

for all injective mappings $\varphi\colon [\ell] \to [r]$. By applying Lemma 3.20 we derive that all edges $\{i_1, \ldots, i_k\}$ have weight $d \pm \varepsilon$ and, therefore, $d(V_{i_1}, \ldots, V_{i_k}) = d \pm \varepsilon$ which finishes the proof of Lemma 3.22. $\square$

3.3.3 DISC_d implies $\mathrm{HCL}_{d,F,\alpha}$

In this section we deduce $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}$ from DISC_d by proving the following lemma.

Fact 3.24. *For every integer $k \geq 2$, every $d > 0$, every linear k-uniform hypergraph F with at least one edge and $V(F) = [\ell]$ for some integer ℓ, and every vector $\boldsymbol{\alpha} \in (0,1]^\ell$, there exist $\delta > 0$ and n_0 such that the following is true. If H is k-uniform hypergraph on $n \geq n_0$ vertices that satisfies $\mathrm{DISC}_d(\delta)$, then H satisfies $\mathrm{HCL}_{d,F,\boldsymbol{\alpha}}(\varepsilon)$.*

Proof. The fact is a simple consequence of the counting lemma, Lemma 2.6. Indeed for given k, $d > 0$, F, $\boldsymbol{\alpha} \in (0,1]^\ell$, and $\varepsilon > 0$, set δ to be sufficiently small, so that $\mathrm{DISC}_d(\delta)$ implies $\mathrm{DISC}_{d,k}(\delta')$ (see Theorem 3.4) for $\delta' = (\delta_{\mathrm{CL}} d \min_{i\in\ell} \alpha_i)^k$, where δ_{CL} is given by Lemma 2.6 applied for F and $\gamma_{\mathrm{CL}} = \varepsilon/2$ and we may assume $\delta_{\mathrm{CL}} \leq \varepsilon/2$. Let n_0 be sufficiently large and H be a k-uniform hypergraph on $n \geq n_0$ vertices which satisfies $\mathrm{DISC}_d(\delta)$.

Let $U_1, \ldots, U_\ell \subseteq V(H)$ with $|U_i| = \lfloor \alpha_i n \rfloor$ be pairwise disjoint sets. We consider the induced ℓ-partite k-uniform hypergraph $H[U_1, \ldots, U_\ell]$. Since H satisfies $\mathrm{DISC}_d(\delta)$, by

Theorem 3.4 we infer that H satisfies $\mathrm{DISC}_{d,k}(\delta')$. Moreover, since $(\delta')^{1/k}/\min_{i\in[\ell]}\alpha_i \leq \delta_{\mathrm{CL}}$ we have that $(U_{i_1},\dots,U_{i_k})$ is δ_{CL}-regular with density $d \pm \delta_{\mathrm{CL}}$ for every choice $1 \leq i_i < \dots < i_k \leq \ell$. Consequently, Lemma 2.6 implies

$$N_F(U_1,\dots,U_\ell) = (d^{e(F)} \pm (\delta_{\mathrm{CL}} + \gamma_{\mathrm{CL}})) \prod_{i\in[\ell]} |U_i| = d^{e(F)} \prod_{i\in[\ell]} |U_i| \pm \varepsilon n^\ell ,$$

which concludes the proof of the fact. □

3.4 Proof of Theorem 3.4

This section concerns the proof of Theorem 3.4. We have to show that for $k \geq \ell \geq 2$, every (ℓ,k)-function τ, and every $d > 0$ the properties DISC_d, $\mathrm{DISC}_{d,\ell}$, and $\mathrm{DISC}_{d,\tau}$ are equivalent. The equivalence will follow from the implication $\mathrm{DISC}_d \Longrightarrow \mathrm{DISC}_{d,\ell+1}$ (Fact 3.25), which holds for every $\ell = 1,\dots,k-1$, and the equivalence $\mathrm{DISC}_{d,k} \Longleftrightarrow \mathrm{DISC}_{d,\tau}$ (Fact 3.27 and Fact 3.29), which holds for every $\ell = 1,\dots,k$ and every (ℓ,k)-function τ. Theorem 3.4 then follows, since Fact 3.25 applied for all $\ell = 1,\dots,k-1$ gives

$$\mathrm{DISC}_d = \mathrm{DISC}_{d,1} \Rightarrow \dots \Rightarrow \mathrm{DISC}_{d,\ell} \Rightarrow \mathrm{DISC}_{d,\ell+1} \Rightarrow \dots \Rightarrow \mathrm{DISC}_{d,k}$$

and Fact 3.29 applied for the unique $(1,k)$-function $\tau = (1)$ gives

$$\mathrm{DISC}_{d,k} \Rightarrow \mathrm{DISC}_{d,(1)} = \mathrm{DISC}_d .$$

Finally, due to Fact 3.27 and Fact 3.29 we have

$$\mathrm{DISC}_{d,k} \Longleftrightarrow \mathrm{DISC}_{d,\tau}$$

for every $\ell = 1,\dots,k$ and every (ℓ,k)-function τ. We prove Fact 3.25, Fact 3.27, and Fact 3.29 in the next section.

3.4.1 Equivalence of different versions of DISC

We first deduce $\mathrm{DISC}_{d,\ell+1}$ from $\mathrm{DISC}_{d,\ell}$ in a straightforward way.

Fact 3.25. *For all integers $1 \leq \ell < k$, every $d > 0$, and every $\varepsilon > 0$ the following holds. If H is a k-uniform hypergraph that satisfies $\mathrm{DISC}_{d,\ell}(\varepsilon/3)$, then H satisfies $\mathrm{DISC}_{d,\ell+1}(\varepsilon)$.*

Proof. Let $U_1,\dots,U_{\ell+1} \subset V(H)$ be pairwise disjoint sets. Then

$$\begin{aligned}\mathrm{vol}(U_1,\dots,U_{\ell-1},U_\ell,U_{\ell+1}) = {} & \mathrm{vol}(U_1,\dots,U_{\ell-1},U_\ell \dot\cup U_{\ell+1}) \\ & - \mathrm{vol}(U_1,\dots,U_{\ell-1},U_\ell) - \mathrm{vol}(U_1,\dots,U_{\ell-1},U_{\ell+1}).\end{aligned}$$

and

$$\begin{aligned}e(U_1,\dots,U_{\ell-1},U_\ell,U_{\ell+1}) = e(U_1,\dots,U_{\ell-1},U_\ell\dot\cup U_{\ell+1})\\ - e(U_1,\dots,U_{\ell-1},U_\ell) - e(U_1,\dots,U_{\ell-1},U_{\ell+1}).\end{aligned}$$

Since H satisfies $\mathrm{DISC}_{d,\ell}(\varepsilon/3)$ we have

$$e(U_1,\dots,U_{\ell-1},X) = d\mathrm{vol}(U_1,\dots,U_{\ell-1},X) \pm \varepsilon n^k/3$$

for all $X \in \{U_\ell, U_{\ell+1}, U_\ell\dot\cup U_{\ell+1}\}$ and, consequently

$$e(U_1,\dots,U_\ell,U_{\ell+1}) = d\mathrm{vol}(U_1,\dots,U_\ell,U_{\ell+1}) \pm \varepsilon n^k.$$

□

We continue with the following observation, which is a direct consequence of the principle of inclusion and exclusion.

Fact 3.26. *Let t, ℓ, and k be positive integers with $t+\ell \le k+1$ and let $\tau \in T(\ell,k)$ be an (ℓ,k)-function with $\tau(\ell)=t$. Let τ' be the $(\ell+t-1,k)$-function given by*

$$\tau'(i) = \begin{cases}\tau(i) & \text{if } i<\ell\\ 1 & \text{if } i \ge \ell.\end{cases}$$

Then for every k-uniform hypergraph H and all $\ell+t-1$ pairwise disjoint sets

$$U_1,\dots,U_{\ell-1},U_\ell^1,\dots U_\ell^t \subset V(H)$$

we have

$$e_{\tau'}(U_1,\dots,U_{\ell-1},U_\ell^1,\dots,U_\ell^t) = \textstyle\sum_{\varnothing\neq J\subseteq[t]}(-1)^{t-|J|}e_\tau(U_1,\dots,U_{\ell-1},\bigcup_{j\in J}U_\ell^j)\,.$$

Proof. Let $K \subset \dot\bigcup_{j\in[\ell-1]}U_j\dot\cup\dot\bigcup_{j\in[t]}U_\ell^j$ be a set of size k such that $K\cap U_i = \tau(i)$ for all $i<\ell$ and let $I_K = \{i\colon\ |K\cap U_\ell^i|>0\}$. Note that K appears in $e_{\tau'}(U_1,\dots,U_{\ell-1},U_\ell^1,\dots,U_\ell^t)$ if and only if $|I_K|=t$. Moreover, the contribution of K to the right-hand side is

$$\sum_{I_K\subseteq J\subseteq[t]}(-1)^{t-|J|} = \sum_{j=0}^{t-|I_K|}\binom{t-|I_K|}{j}(-1)^{t-(|I_K|+j)} = \begin{cases}1 & \text{if } |I_K|=t\\ 0 & \text{otherwise.}\end{cases}$$

□

Fact 3.27. *For all integers $1\le\ell\le k$, every $d>0$, every (ℓ,k)-function τ, and every $\varepsilon>0$ the following holds. If H is a k-uniform hypergraph that satisfies $\mathrm{DISC}_{d,\tau}(\varepsilon/2^{k^2/2})$, then H satisfies $\mathrm{DISC}_{d,k}(\varepsilon)$.*

Proof. Recall first that $\mathrm{DISC}_{d,k}(\varepsilon) = \mathrm{DISC}_{d,\sigma}(\varepsilon)$ if σ is the everywhere 1-function or equivalently the unique (k,k)-function. For a given τ we call $|\{i\colon\ \tau(i)\ge 2\}|$ the defect of τ. Since the everywhere 1-function σ is the only (ℓ,k)-function, for any ℓ, with defect 0, the fact follows from at most $\lfloor k/2\rfloor$ applications of the following claim. $\square$

Claim 3.28. *Suppose τ is an (ℓ,k)-function with defect $s \geq 1$. Then there is a τ' with defect $s-1$ such that if H satisfies $\mathrm{DISC}_{d,\tau}(\varepsilon/2^k)$, then H satisfies $\mathrm{DISC}_{d,\tau'}(\varepsilon)$.*

Proof. Claim 3.28 follows from Fact 3.26. For a given $\tau \in T(\ell,k)$ with defect $s \geq 1$ we may assume without loss of generality that $\tau(\ell) = t \geq 2$. We define the $(\ell+t-1,k)$-function τ' by

$$\tau'(i) = \begin{cases} \tau(i) & \text{if } i < \ell \\ 1 & \text{if } i \geq \ell. \end{cases} \tag{3.24}$$

Then τ' has defect $s-1$ and from Fact 3.26 we infer

$$e_{\tau'}(U_1,\dots,U_{\ell-1},U_\ell^1,\dots,U_\ell^t) = \textstyle\sum_{\varnothing\neq J\subseteq[t]}(-1)^{t-|J|}e_\tau(U_1,\dots,U_{\ell-1},\bigcup_{j\in J}U_\ell^j)$$

and

$$\mathrm{vol}_{\tau'}(U_1,\dots,U_{\ell-1},U_\ell^1,\dots,U_\ell^t) = \textstyle\sum_{\varnothing\neq J\subseteq[t]}(-1)^{t-|J|}\mathrm{vol}_\tau(U_1,\dots,U_{\ell-1},\bigcup_{j\in J}U_\ell^j)$$

for any choice of pairwise disjoint sets $U_1,\dots,U_{\ell-1}$, $U_\ell^1,\dots U_\ell^t \subset V(H)$. Since H satisfies $\mathrm{DISC}_{d,\tau}(\varepsilon/2^k)$ we have

$$e_\tau(U_1,\dots,U_{\ell-1},\textstyle\bigcup_{j\in J}U_\ell^j) = d\mathrm{vol}_\tau(U_1,\dots,U_{\ell-1},\bigcup_{j\in J}U_\ell^j) \pm \varepsilon n^k/2^k$$

for all $\varnothing \neq J \subseteq [t]$ and, hence,

$$\begin{aligned} &e_{\tau'}(U_1,\dots,U_{\ell-1},U_\ell^1,\dots,U_\ell^t) \\ &\qquad= \sum_{\varnothing\neq J\subseteq[t]}(-1)^{t-|J|}\Big(d\mathrm{vol}_\tau\Big(U_1,\dots,U_{\ell-1},\bigcup_{j\in J}U_\ell^j\Big) \pm \varepsilon n^k/2^k\Big) \\ &\qquad= d\sum_{\varnothing\neq J\subseteq[t]}(-1)^{t-|J|}\mathrm{vol}_\tau\Big(U_1,\dots,U_{\ell-1},\bigcup_{j\in J}U_\ell^j\Big) \pm 2^{t-k}\varepsilon n^k \\ &\qquad\qquad= d\mathrm{vol}_{\tau'}(U_1,\dots,U_{\ell-1},U_\ell^1,\dots,U_\ell^t) \pm \varepsilon n^k. \end{aligned}$$

$\square$

The last observation in this section reverses the implication of Fact 3.27.

Fact 3.29. *For all integers $1 \leq \ell \leq k$, every $d > 0$, every (ℓ,k)-function τ, and every $\varepsilon > 0$ there is an n_0 such that the following holds. If H is a k-uniform hypergraph on $n \geq n_0$ vertices that satisfies $\mathrm{DISC}_{d,k}(\varepsilon/3^{k^2})$, then H satisfies $\mathrm{DISC}_{d,\tau}(\varepsilon)$.*

Proof. We choose n_0 sufficiently large and by induction on $\ell = k,\dots,1$ we prove that if H satisfies $\mathrm{DISC}_{d,k}(\varepsilon/3^{(k-\ell)k})$ then H also satisfies $\mathrm{DISC}_{d,\tau}(\varepsilon)$ for an arbitrary (ℓ,k)-function τ.

For $\ell = k$ there is only one (ℓ, k)-function τ which is the everywhere 1-function. Then $\mathrm{DISC}_{d,\tau}(\varepsilon) = \mathrm{DISC}_{d,k}(\varepsilon)$ and the implication is obviously true.

So suppose by induction that for every $(\ell+1, k)$-function τ' every k-uniform hypergraph H on n vertices with the property $\mathrm{DISC}_{d,k}(\varepsilon/3^{(k-\ell)k})$ also satisfies $\mathrm{DISC}_{d,\tau'}(\varepsilon/3^k)$.

Let τ be an arbitrary (ℓ, k)-function and let $U_1, \ldots, U_\ell \subseteq V(H)$ be pairwise disjoint sets. Without loss of generality we assume that $\tau(\ell) = t \geq 2$ and we define an $(\ell+1, k)$-function τ' by

$$\tau'(i) = \begin{cases} \tau(i) & \text{if } i < \ell \\ \tau(i) - 1 & \text{if } i = \ell \\ 1 & \text{if } i = \ell + 1. \end{cases} \tag{3.25}$$

Further let $\mathcal{P}(U_\ell)$ be the family of all ordered bipartitions of U_ℓ into two equitable sets, i.e. all pairs (W_1, W_2) with $U_\ell = W_1 \dot\cup W_2$ and $|W_1| = \lfloor |U_\ell|/2 \rfloor = w$. Then

$$\mathrm{vol}_{\tau'}(U_1, \ldots, U_{\ell-1}, W_1, W_2) = \binom{w}{t-1}(|U_\ell| - w) \prod_{i \in [\ell-1]} \binom{|U_i|}{\tau(i)}$$

holds for all bipartitions $(W_1, W_2) \in \mathcal{P}(U_\ell)$. Since H satisfies $\mathrm{DISC}_{d,\tau'}(\varepsilon/3^k)$ we have

$$e_{\tau'}(U_1, \ldots, U_{\ell-1}, W_1, W_2) = d\mathrm{vol}_{\tau'}(U_1, \ldots, U_{\ell-1}, W_1, W_2) \pm \varepsilon n^k/3^k\,.$$

Summing over all bipartitions in $\mathcal{P}(U_\ell)$ every edge in $E_\tau(U_1, \ldots, U_\ell)$ is counted exactly $t\binom{|U_\ell|-t}{w-(t-1)}$ times. Thus, we infer

$$\begin{aligned} e_\tau(U_1, \ldots, U_\ell) &= \frac{1}{t\binom{|U_\ell|-t}{w-(t-1)}} \sum_{(W_1,W_2) \in \mathcal{P}(U_\ell)} e_{\tau'}(U_1, \ldots, U_{\ell-1}, W_1, W_2) \\ &= \frac{|\mathcal{P}(U_\ell)|}{t\binom{|U_\ell|-t}{w-(t-1)}} \left(d\binom{w}{t-1}(|U_\ell| - w) \prod_{i \in [\ell-1]} \binom{|U_i|}{\tau(i)} \pm \varepsilon n^k/3^k \right). \end{aligned}$$

With $|\mathcal{P}(U_\ell)| = \binom{|U_\ell|}{w}$ and

$$\frac{|\mathcal{P}(U_\ell)|}{t\binom{|U_\ell|-t}{w-(t-1)}} \binom{w}{t-1}(|U_\ell| - w) = \binom{|U_\ell|}{t}$$

and since $|\mathcal{P}(U_\ell)| \leq 3^k t\binom{|U_\ell|-t}{w-(t-1)}$ we obtain

$$e_\tau(U_1, \ldots, U_\ell) = d \prod_{i \in [\ell]} \binom{|U(i)|}{\tau(i)} \pm \varepsilon n^k.$$

□

3.5 An application: a strong refutation algorithm

In this section we consider a strong refutation algorithm for random k-SAT. It is well-known that for $p \gg n^{1-k}$ with high probability a random formula $F \in \mathcal{F}_k(n,p)$ is not satisfiable. However, there are no efficient refutation algorithms known. We are interested in deterministic, polynomial time algorithms which w.h.p. reject a k-SAT formula from $\mathcal{F}_k(n,p)$ for $p \gg n^{1-k}$, but which never reject a satisfiable formula. Recall, that an algorithm is a strong refutation algorithm if w.h.p. for $F \in \mathcal{F}_k(n,p)$ it approximates $\mathrm{unsat}(F)$ by a factor of $(1-\varepsilon)$ and never outputs a number bigger than $\mathrm{unsat}(F)$, where $\mathrm{unsat}(F)$ is the minimum number of unsatisfied clauses in F over all possible assignments.

3.5.1 Proof of Theorem 1.6

Our work is based on results on quasi-random hypergraphs found in [CHPS] and discussed earlier in this chapter. Here we will use some partite notions of DISC and DEV, called PDISC and PDEV.

With every $F \in \mathcal{F}_k(n,p)$ we will associate a k-partite, k-uniform hypergraph H_F in the following way. Let X_n be the variables of F. We denote by $V_n = \{x_1, \bar{x}_1, \ldots x_n, \bar{x}_n\}$ the literals of F and let $V(H_F)$ consist of k copies of V_n, i.e., $V(H_F) = V_n \times [k]$. Moreover, the edges of H_F correspond to the clauses of F, i.e., $\{(v_1,1), \ldots, (v_k,k)\}$ is an edge of H_F if and only if $v_1 \vee \ldots \vee v_k$ is a clause in F. Clearly, this defines a bijection between all k-SAT formulas on X_n (the clauses are defined as ordered k-tuples of literals) and all k-partite, k-uniform hypergraphs with vertex classes $(V_n \times \{1\}) \dot\cup \ldots \dot\cup (V_n \times \{k\})$. Moreover, it is well-known that $\mathrm{unsat}(F)$ is related to the *discrepancy* of H_F.

Definition 3.30. *Suppose $H = (V_1 \dot\cup \ldots \dot\cup V_k, E)$ is a k-partite, k-uniform hypergraph with vertex classes of size N and density $p = |E|/N^k$. For $\varepsilon > 0$ we say that H satisfies* PDISC(ε) *if for all subsets $U_1 \subseteq V_1, \ldots, U_k \subseteq V_k$*

$$\big|e_H(U_1, \ldots, U_k) - p|U_1| \cdots |U_k|\big| < \varepsilon p N^k.$$

Note that every assignment β of the variables $\{x_1, \ldots, x_n\}$ corresponds to a bipartition of each $V_n \times \{i\}$ into equally large sets of literals $U_i = \{(v,i) \in V_n \times \{i\} \colon \beta(v) = 0\}$ and $(V_n \times \{i\}) \setminus U_i$. Furthermore, the number of clauses not satisfied by β corresponds to the number of edges spanned by $U_1 \dot\cup \ldots \dot\cup U_k$. This observation yields the following.

Fact 3.31. *If H_F satisfies* PDISC(ε) *then* $\mathrm{unsat}(F) \geq (2^{-k} - \varepsilon)|F|$. □

For a k-partite, k-uniform hypergraph H with vertex partition $V_1 \dot\cup \ldots \dot\cup V_k$ and density p let $w_H \colon \prod_{i \in [k]} V_i \to [-1,1]$ be the function defined by $w_H(e) = 1 - p$ if $e \in E(H)$ and $w_H(e) = -p$ otherwise.

Definition 3.32. *Suppose $H = (V_1 \dot\cup \ldots \dot\cup V_k, E)$ is a k-partite, k-uniform hypergraph with vertex classes of size N and density $p = |E|/N^k$. For $\varepsilon > 0$ we say that H satisfies*

PDEV(ε) *if for* $\mathcal{V} = (V_1, \dots, V_k)$

$$|\sum_{\varphi\in\mathrm{Hom}(M_k,\mathcal{V})}\prod_{e\in E(M_k)} w_H(\varphi(e))| \le \varepsilon p^{2^k} N^{k2^{k-1}}.$$

While the property PDISC cannot be naively verified in polynomial time, it was shown for dense hypergraphs, see Theorem 1.3, that the property DISC is equivalent to an efficiently verifiable property, called DEV, which measures the distribution of the homomorphisms of a certain hypergraph M_k. We will show that the implication "*hypergraphs satisfying* PDEV *must satisfy* PDISC" still holds for hypergraphs of density $p = o(1)$. Theorem 1.6 then follows from the observation that w.h.p. H_F satisfies PDEV if $p \gg n^{-(k-1)/2}$.

Lemma 3.33. *For every $k \ge 3$ and $\varepsilon > 0$ there exists n_0 such that for all $N \ge n_0$ the following holds. Suppose $H = (V_1\dot\cup\dots\dot\cup V_k, E)$ is a k-partite, k-uniform hypergraph with vertex classes of size N. If H satisfies* PDEV(ε^{2^k})*, then H also satisfies* PDISC(ε)*.*

Lemma 3.34. *For any $k \ge 3$, $\varepsilon > 0$, and $o(1) = p(n) \gg n^{-(k-1)/2}$ we have*

$$\lim_{n\to\infty} \mathbb{P}(H_F \textit{ satisfies } \mathrm{PDEV}(\varepsilon)) = 1$$

for $F \in \mathcal{F}_k(n,p)$.

With these lemmas at hand, whose proofs are presented in the next section, we can show the existence of the desired algorithm.

Proof of Theorem 1.6. The property PDEV(δ) can be verified in $O(N^{k2^{k-1}})$ time. Lemma 3.33 combined with Fact 3.31 shows that the algorithm $\mathcal{A}$, which for a k-SAT formula F outputs 0 if H_F fails to satisfy PDEV(ε^{2^k}) and outputs $(2^{-k} - \varepsilon)|F|$ otherwise, fulfills part (i) of Definition 1.5. Moreover, Lemma 3.34 combined with (1.1) shows that the algorithm $\mathcal{A}$ also satisfies part (ii) of Definition 1.5 for $F \in \mathcal{F}_k(n,p)$ with $p \gg n^{-(k-1)/2}$. $\square$

3.5.2 Proofs of Lemmas 3.33 and 3.34

Proof of Lemma 3.33. Recall, that the hypergraph M_k is the k-uniform, k-partite hypergraph, which we obtained through fixing a single hyperedge and the process of doubling, see Section 3.1.1, and the sets $X_1^j, \dots, X_k^j$ denote the partition classes of $M_j^{(k)}$ (here the doubling was applied j times). Further, for a k-tuple of vertex sets $\mathcal{V} = (V_1, \dots, V_k)$ we denote by Hom($M_j, \mathcal{V}$) all functions $\varphi\colon \bigcup_{i\in[k]} X_i^j \to \bigcup_{i\in[k]} V_i$ with $\varphi(X_i^j) \subseteq V_i$ for all $i \in [k]$. In other words, Hom($M_j, \mathcal{V}$) is the set of all (partition respecting) homomorphisms from M_j to the complete k-partite, k-uniform hypergraph on the partition classes $V_1\dot\cup\dots\dot\cup V_k$.

The proof follows the lines of Lemma 3.11. Let $H = (V_1\dot\cup\dots\dot\cup V_k, E)$ be a k-partite, k-uniform hypergraph with vertex classes of size N and density p, which satisfies DEV(ε^{2^k}).

Let $U_1 \subseteq V_1, \ldots, U_k \subseteq V_k$. We show

$$|e_H(U_1, \ldots, U_k) - p \prod_{i \in [k]} |U_i|| \leq \varepsilon p N^k.$$

Set $\mathcal{U}_i = (V_1, \ldots, V_i, U_{i+1}, \ldots, U_k)$ and for $j \in \{0, \ldots, k\}$ let

$$f_H(M_j, \mathcal{U}_j) = \textstyle\sum_{\varphi \in \mathrm{Hom}(M_j, \mathcal{U}_j)} \prod_{e \in E(M_j)} w_H(\varphi(e)). \tag{3.26}$$

Note that by definition $f_H(M_0, \mathcal{U}_0) = e_H(U_1, \ldots, U_k) - p \prod_{i \in [k]} |U_i|$ and, since H satisfies $\mathrm{DEV}(\varepsilon^{2^k})$, $f_H(M_k, \mathcal{U}_k) \leq \varepsilon^{2^k} p^{2^k} N^{k2^{k-1}}$. On the other hand, we can rewrite (3.26) in the following way. For an arbitrary ordering $\boldsymbol{x} = (x_1, \ldots, x_{2^j})$ of the vertices in the j-th vertex class $X_{j+1}(M_j)$ of M_j, we fix the image of $\boldsymbol{x}$ to be $\boldsymbol{v} = (v_1, \ldots, v_{2^j}) \in U_{j+1}^{2^j}$, i.e. map x_i to v_i for all $i \in [2^j]$, and extend this choice to a homomorphism $\varphi \in \mathrm{Hom}(M_j, \mathcal{U}_j)$. Consequently,

$$f_H(M_j, \mathcal{U}_j) = \textstyle\sum_{\boldsymbol{v} \in U_{j+1}^{2^j}} \sum_{\substack{\varphi \in \mathrm{Hom}(M_j, \mathcal{U}_j) \\ \varphi(\boldsymbol{x}) = \boldsymbol{v}}} \prod_{e \in E(M_j)} w_H(\varphi(e)). \tag{3.27}$$

Recall, that $M_{j+1} = \mathrm{db}_{j+1}(M_j)$ arises from M_j by fixing the $(j+1)$-st vertex class $X_{j+1}(M_j)$ of M_j and "doubling" all the edges together with the remaining vertices. Thus, applying the Cauchy-Schwarz inequality to $f_H(M_j, \mathcal{U}_j)$ (to the form stated in (3.27)), we obtain $f_H(M_j, \mathcal{U}_j)^2 \leq |U_{j+1}|^{2^j} f_H(M_{j+1}, \mathcal{U}_{j+1})$ for every $j \in \{0, \ldots, k-1\}$. Applying this inductively for $j = 0, \ldots, k-1$ we obtain

$$\left|f_H(M_0, \mathcal{U}_0)\right|^{2^k} \leq \textstyle\prod_{i \in [k]} |U_i|^{2^{k-1}} \left|f_H(M_k, \mathcal{U}_k)\right| \leq \varepsilon^{2^k} p^{2^k} N^{k2^k}.$$

Consequently, $|e(U_1, \ldots, U_k) - p \prod_{i \in [k]} |U_i|| = |f_H(M_0, \mathcal{U}_0)| \leq \varepsilon p N^k$. □

Proof of Lemma 3.34. For $k \geq 3$ and $\varepsilon > 0$ let $o(1) = p \gg n^{-(k-1)/2}$. Set $\delta = \varepsilon/(12 \cdot 2^{2^k})$ and let $\mathcal{M}$ be the set of all spanning subgraphs of M_k. Let $\mathcal{B}$ be the set of all labeled k-uniform hypergraphs B on $v_B < k2^{k-1}$ vertices such that there is a surjective homomorphism from M_k to B. For a k-partite hypergraph C, let X_C be the random variable denoting the number of labeled partition respecting copies of C in H_F with $F \in \mathcal{F}_k(n, p)$.

Claim 3.35. *With high probability we have*

(*a*) $X_A = (1 \pm \delta)\mathbb{E}(X_A)$ *for all* $A \in \mathcal{M}$*, and*

(*b*) $\sum_{B \in \mathcal{B}} X_B < \delta X_{M_k}$.

Proof. For part (*a*) we note that, since every vertex of M_k is contained in precisely two edges, the hypergraph M_k is balanced, i.e., $e_{M_k}/v_{M_k} = 2^k/k2^{k-1} = 2/k \geq e_A/v_A$ for all (not necessarily spanning) subhypergraphs $A \subseteq M_k$. Moreover, it is easy to check that for the p considered here, we have $\mathbb{E}(X_A) \geq \mathbb{E}(X_{M_k}) \to \infty$ for every $A \in \mathcal{M}$. Hence, part (a) follows easily from Chebyshev's inequality applied in a similar way as, e.g., in [JŁR00, Theorem 3.4].

Due to part (a), it suffices to show that w.h.p. $X_B \leq \delta p^{2^k}(2n)^{k2^{k-1}}/(2|\mathcal{B}|)$ for every $B \in \mathcal{B}$ to conclude assertion (b). Let $B \in \mathcal{B}$ and set $q = 2^k - e_B$ and $r = k2^{k-1} - v_B$. Hence, $p^{2^k}(2n)^{k2^{k-1}} = (1-o(1))(p(2n)^{r/q})^q\mathbb{E}(X_B)$ and below we will show that $r \geq (k-1)q/2$, which due to our choice of p yields that $\mathbb{E}(X_B) = o(p^{2^k}(2n)^{k2^{k-1}})$ and assertion (b) follows from Markov's inequality.

Let $\varphi\colon M_k \to B$ be a surjective homomorphism. For $e \in E(B)$ let $\{f_1, \ldots, f_m\} = \varphi^{-1}(e) \subseteq E(M_k)$. Fix f_1 and call f_i, $i \neq 1$, a *lost edge* and any vertex $v \in f_i \setminus f_1$ a *lost vertex.* There are q lost edges and every lost edge contains at least $(k-1)$ lost vertices (f_i and f_1 intersect in at most one vertex, since M_k is a linear hypergraph). On the other hand, the number of lost vertices is at most r and every lost vertex is contained in at most two (lost) edges. Thus, by double counting we have $q(k-1) \leq 2r$. $\square$

We deduce Lemma 3.34 from Claim 3.35. Let $\mathrm{Inj}(M_k, \mathcal{V}) \subseteq \mathrm{Hom}(M_k, \mathcal{V})$ be the set of all injective mappings $\varphi \in \mathrm{Hom}(M_k, \mathcal{V})$. Thus, every $\varphi \in \mathrm{Inj}(M_k, \mathcal{V})$ corresponds to an $\widetilde{A} \subseteq H$ which is a labeled copy of some $A \in \mathcal{M}$ in H, whereas any $\varphi \in \mathrm{Hom}(M_k, \mathcal{V}) \setminus \mathrm{Inj}(M_k, \mathcal{V})$ corresponds to a $\widetilde{B} \subset H$ which is labeled copy of a hypergraph $B \in \mathcal{B}$. Let $\hat{X}_A$ be the number of induced copies of A. Since $p = o(1)$ we have w.h.p. $\hat{X}_A = (1-o(1))X_A$ and $(1-p)^{k2^{k-1}} \geq 1 - \delta$. Since w.h.p. $e(H_F)/(2n)^k = (1+o(1))p$, part (a) of Claim 3.35 yields w.h.p.

$$\begin{aligned}\sum_{\varphi \in \mathrm{Inj}(M_k,\mathcal{V})} \prod_{e \in E(M_k)} w_H(\varphi(e)) &= (1-o(1)) \sum_{A \in \mathcal{M}} (1-p)^{e_A}(-p)^{2^k - e_A}\hat{X}_A \\ &= p^{2^k} \sum_{A \in \mathcal{M}} (1 \pm 3\delta)(-1)^{2^k - e_A}(2n)^{k2^{k-1}} \leq 6\delta 2^{2^k} p^{2^k}(2n)^{k2^{k-1}} \leq \tfrac{\varepsilon}{2} p^{2^k}(2n)^{k2^{k-1}}.\end{aligned}$$

Moreover, due to parts (a) and (b) of the Claim 3.35 w.h.p. we can bound

$$\left| \sum_{\varphi \in \mathrm{Hom}(M_k,\mathcal{V}) \setminus \mathrm{Inj}(M_k,\mathcal{V})} \prod_{e \in E(M_k)} w_H(\varphi(e)) \right| \leq \sum_{B \in \mathcal{B}} X_B \leq \delta X_{M_k} \leq \tfrac{\varepsilon}{2} p^{2^k}(2n)^{k2^{k-1}}.$$

Thus for $F \in \mathcal{F}_k(p,n)$ the hypergraph H_F satisfies w.h.p. $\mathrm{PDEV}(\varepsilon)$. $\square$

3.6 Concluding remarks

3.6.1 Extension of P_3

For Theorem 1.3 we extended properties P_1, P_2, P_4, P_6, and P_7. While the extension of P_5 is straightforward and its equivalence to DISC_d follows along the lines of [Yus08], we did not find an interesting generalization of P_3 for k-graphs and leave this open.

3.6.2 Uniform edge distribution with respect to i-sets

We studied quasi-random properties equivalent to uniform edge distribution of k-uniform hypergraphs with respect to large vertex sets. A natural generalization concerns the edge distribution with respect to large subsets of i-tuples.

i-DISC$_d(\varepsilon)$ We say a k-uniform hypergraph $H = (V, E)$ on n vertices has i-DISC$_d(\varepsilon)$ for $1 \leq i \leq k-1$, d, $\varepsilon > 0$, if

$$|E(H) \cap \mathcal{K}_k(G^{(i)})| = d|\mathcal{K}_k(G^{(i)})| \pm \varepsilon n^k ,$$

for any i-uniform hypergraph $G^{(i)}$ with vertex set V, where $\mathcal{K}_k(G^{(i)})$ denotes the set of all k-sets K in $[V]^k$ which span a copy of $K_k^{(i)}$ (the complete i-graph on k vertices) in $G^{(i)}$.

Clearly, i-DISC$_d$ for $i = 1$ coincides with DISC$_d$ and for $i = k-1$ this is the central concept of quasi-randomness studied in [KRS02]. The general notion i-DISC$_d$ was first studied by Frankl and Rödl [FR92] and Chung [Chu90, Chu91]. We believe that Theorem 1.3 can be extended for general i. As 1-DISC$_d$ is characterized by the subgraph frequencies of linear k-uniform hypergraphs, i-DISC$_d$ is closely related to the appearance of partial Steiner $(i+1, k)$-systems, i.e., k-uniform hypergraphs for which every two hyperedges intersect in at most i vertices. In this context the natural generalization of the "doubling" operation from Section 3.1.1 seems to be the following. Let A be a k-partite k-uniform hypergraph with vertex classes $X_1, \ldots, X_k$ and let $I \in \binom{[k]}{i}$ be an i-set, then the doubling $\mathrm{db}_I(A)$ of A is obtained by taking two copies of A and identifying the vertices in the classes X_i for all $i \in I$. Again starting with a single hyperedge and applying consecutively db_I for every $I \in \binom{[k]}{i}$ (in some arbitrary order) we will get a k-partite k-uniform hypergraph, which seems likely to be of similar importance for i-DISC$_d$ as M had in Theorem 1.3. In fact, for $i = k-1$, this way we obtain the k-uniform hypergraph of the octahedron $K^{(k)}_{2,\ldots,2}$ which was already studied in connection with $(k-1)$-DISC$_d$ in [CG90, KRS02].

A related line of research concerns the connection to extensions of Szemerédi's regularity lemma. While there is a regularity lemma which decomposes any given k-uniform hypergraph into relatively few "blocks" such that most of them satisfy a k-partite version 1-DISC$_d$ (i.e., DISC$_{d,k}$), for $i \geq 2$ the notion of i-DISC seems too strong and likely no regularity lemma compatible for this notion exists. Instead, one needs to work with "relative" versions of i-DISC. For $i = k-1$, this notion of quasi-randomness was introduced in the work on hypergraph regularity by Rödl et al. [FR02, RS04] and Gowers [Gow06, Gow07], and for $k = 3$ the equivalence was studied in [NPRS09]. It would be interesting to further investigate those connections for general i.

3.6.3 Extension of Corollary 1.4

In Corollary 1.4 we showed that for every $k \geq 2$ the complete graph K_k and the line graph of the k-dimensional hypercube $M(k)$ (which alternatively can be obtained from the k-uniform hypergraph M_k by replacing every hyperedge of M_k with a graph clique

K_k) is a quasi-random pair. The construction of $M(k)$ can be easily extended from cliques to arbitrary graphs F. For a graph F with vertex set $[k]$ let $M(F)$ be the graph obtained from the k-uniform hypergraph M_k with vertex classes $X_1, \ldots, X_k$ by replacing every hyperedge by a copy of F such that the vertex representing vertex $i \in [k] = V(F)$ lies in X_i. It seems possible that $(F, M(F))$ is a quasi-random pair for every graph F. Indeed the following observation supports this belief.

While the notion of quasi-random pairs is closely related to the property MIN_d, we may also consider the following version of DEV_d for graphs.

$\mathrm{DEV}_{d,F}(\varepsilon)$ We say a graph $G = (V, E)$ on n vertices has $\mathrm{DEV}_{d,F}(\varepsilon)$ for a graph F with vertex set $[k]$ and d, $\varepsilon > 0$, if

$$\left| \sum_{\widetilde{M}} \prod_{\widetilde{F} \subseteq \widetilde{M}} \left(\left(\prod_{e \in E(\widetilde{F})} \mathbf{1}_E(e) \right) - d^{e(F)} \right) \right| \leq \varepsilon n^{k 2^{k-1}},$$

where the sum runs over all copies $\widetilde{M}$ of $M(F)$ in the complete graph K_V on vertex set V and the outer product runs over the 2^k copies $\widetilde{F}$ of F (corresponding to the hyperedges of M_k).

Following closely the lines of the proof of Lemma 3.11 it can be shown that for every $d > 0$ and every graph F with at least one edge, a graph G satisfying $\mathrm{DEV}_{d,F}(\varepsilon)$ also satisfies the assumptions of Theorem 1.2 and consequently such graphs are quasi-random with density d.

3.6.4 Algorithmic considerations

Since DEV_d, MIN_d, and MDEG_d can be easily checked in polynomial time, in fact in $O(n^{k2^{k-1}})$, we obtain by Theorem 1.3 an efficient algorithm which can approximately check whether a given k-uniform hypergraph has DISC_d. More precisely, for any given d and $\varepsilon > 0$ there exists some positive $\varepsilon' < \varepsilon$ such that the algorithm can distinguish in polynomial time, whether a given k-uniform hypergraph H satisfies $\mathrm{DISC}_d(\varepsilon)$ or fails to satisfy $\mathrm{DISC}_d(\varepsilon')$. In some sense we cannot hope for an efficient algorithm, which decides $\mathrm{DISC}_d(\varepsilon)$ precisely, since it was shown in [ADL+94] that deciding $\mathrm{DISC}_d(\varepsilon)$ for graphs is co-NP complete.

Likely such an approximation algorithm can be used for an algorithmic version of the weak hypergraph regularity lemma, Theorem 2.7. Such an algorithm would find an ε-regular partition in $O(n^{k2^{k-1}})$. However, a more efficient algorithm, with running time $O(n^{2k-1} \log^2 n)$ was found by Czygrinow and Rödl [CR00].

In the graph case, the best known deterministic algorithm for constructing an ε-regular partition runs in the optimal time $O(n^2)$ [KRT03]. The first $O(n)$ time randomized algorithm that constructs a regular partition of an n-vertex graph with high probability was found by Frieze and Kannan [FK99]. Recently, Fischer, Matsliah and Shapira [FMS07] gave another randomized algorithm that finds with high probability an ε-regular partition in *expected* time $O(n)$ with better parameters hidden in big-Oh notation. Furthermore, their algorithm finds with high probability small regular partitions, if such exist. It is likely that using Lemma 3.8 one should be able to extend their result to k-uniform hypergraphs, i.e. to design a randomized algorithm that finds a "minimal" weak ε-regular partition in expected time $O(n)$ with high probability.

The proof of the implication $\mathrm{DEV}_d \Rightarrow \mathrm{DISC}_d$, Lemma 3.11, extends to sparse k-uniform hypergraphs, see Lemma 3.33, i.e., for the case $d = o(1)$ as long as $d \gg n^{-(k-1)/2}$, we obtain a sufficient, efficiently verifiable condition for checking DISC_d for sparse k-uniform hypergraphs. We believe it would be interesting to investigate this problem further. For example, we are not aware of a property which is equivalent to DISC_d as long as $d \gg n^{-k+1}$ and which can be verified in polynomial time.

It would further be interesting to study the sparse case similar to the work done for graphs, see for example [Koh97, CG02, GS05, ACOH$^+$07].

3.6.5 Non forcing pairs

In [CG91] Chung and Graham constructed a family of hypergraphs $(H_n)_{n\in\mathbb{N}}$ that satisfy a generalization of property P_1 for all k-uniform hypergraphs on ℓ vertices, where ℓ is some fixed integer with $2 \leq \ell \leq 2k-1$, but these hypergraphs $(H_n)_{n\in\mathbb{N}}$ fail to satisfy $(k-1)$-$\mathrm{DISC}_{1/2}$. It would be interesting to know whether our generalization of P_1, i.e. ICL, would hold if we restrict counting to only linear hypergraphs on some fixed number of vertices, say ℓ and $k \leq \ell < k2^{k-1}$.

In this section we make a small step in this direction and show that there exists no minimal configuration for 3-graphs with 6 or less vertices. In other words for 3-graphs the 3-graph M from property MIN_d with 8 edges and 12 vertices can not be replaced by a 3-graph on at most 6 vertices. Hence, for every linear 3-graph F on six vertices we have to construct 3-graphs of density $d > 0$ such that they contain the right number of copies of F, but fail to be weak quasi-random, i.e., fail to satisfy DISC_d. There are, up to isomorphism, 6 such 3-graphs F: the one with no edge, with a single edge, with two disjoint edges, with two edges sharing a vertex, the $(6,3)$-configuration (the unique linear 3-graph with 3 edges on six vertices), and the Pasch-configuration (the unique linear 3-graph with 4 edges on six vertices). It is simple to see that for F being one of the first four of those configuration the property that H contains $\sim (2/9)^{e(F)}n^{|V(F)|}$ labeled copies of F does not imply that H has $\mathrm{DISC}_{2/9}$ as for example the complete, 3-partite 3-graph on vertex classes of size $n/3$ shows. Hence we will focus on the $(6,3)$- and the Pasch-configuration.

The $(6,3)$-configuration

We denote by C the $(6,3)$-configuration, which is the 3-graph with $V(C) = [6]$ and $E(C) = \{\{1,2,3\},\{3,4,5\},\{5,6,1\}\}$. We consider the complete 3-partite 3-graph $H = H(\alpha)$ on n vertices with vertex classes V_1, V_2, V_3 such that $|V_1| = |V_2| = (1-\alpha)n/2$ and $|V_3| = \alpha n$ for some $\alpha \in (0, 1/3]$. The density of H is $\frac{3}{2}\alpha(1-\alpha)^2 - o(1)$, while simple calculations show that

$$N_C(H) = \left(\frac{3}{8}\alpha^2(1-\alpha)^4 + o(1)\right) n^6\,,$$

since any copy of C in H must distribute the copies of the vertices $1, 3, 5$ over all three distinct classes, and after fixing the vertex classes of the copies of 1, 3, and 5 the vertex classes of the other three vertices are fixed. Now we need to chose $\alpha > 0$ in such a way that

$$f(\alpha) = \left(\frac{3}{2}\alpha(1-\alpha)^2\right)^3 - \frac{3}{8}\alpha^2(1-\alpha)^4$$

is close to 0, as this would yield that $H = H(\alpha)$ contains the "right" number of copies of C, but clearly H would not satisfy $\mathrm{DISC}_{3\alpha(1-\alpha)^2/2}$. Solving $f(\alpha) = 0$ is equivalent to solving $g(\alpha) = \alpha(1-\alpha)^2$ equals $1/9$. Since $g(0) = 0$ and $g(1/3) = 4/27$, we infer that there exists an $\hat{\alpha} \in (0, 1/3]$ such that $f(\hat{\alpha}) = 0$ (indeed $\hat{\alpha} \approx 0.16$). Hence, $H(\hat{\alpha})$ has the desired properties. Moreover, we obtain other 3-graphs with the same properties (having the right number of copies of C, but failing to have DISC_d) for other densities d, if we consider random subhypergraphs of $H(\hat{\alpha})$.

The Pasch-configuration

Again we will construct a 3-graph H of density d which violates DISC_d, but has $\sim d^4 n^6$ labeled copies of the Pasch-configuration P. For that we first construct a graph G and then consider its triangles to be the hyperedges of H, i.e., $H = \mathcal{K}_3(G)$. Let $G = G(\alpha)$ be the complete, 5-partite graph with vertex classes $V_1 \dot\cup \ldots \dot\cup V_5 = V(G)$ and $|V_1| = |V_2| = |V_3| = |V_4| = (1-\alpha)n/4$ and $|V_5| = \alpha n$. The number of labeled triangles of G satisfies

$$N_{K_3}(G) = \left(\frac{3}{8}(1-\alpha)^3 + \frac{9}{4}(1-\alpha)^2\alpha + o(1)\right) n^3$$

while for the number of labeled $K_{2,2,2}$ in G we have

$$N_{K_{2,2,2}}(G) = \left(\frac{(1-\alpha)^4}{128}(3(1-\alpha)^2 + 126\alpha^2 + 54\alpha(1-\alpha)) + o(1)\right) n^6.$$

As above, we are interested in a solution to

$$\left(\frac{3}{8}(1-\alpha)^3 + \frac{9}{4}(1-\alpha)^2\alpha\right)^4 = \frac{(1-\alpha)^4}{128}(3(1-\alpha)^2 + 126\alpha^2 + 54\alpha(1-\alpha))\,,$$

with $\alpha \in (0, 1/5]$. Since for $\alpha = 0$ the left-hand side is smaller than the right-hand side, while for $\alpha = 1/5$ the inequality switches, and therefore, by the intermediate value theorem, there must be an $\hat{\alpha} \in (0, 1/5]$ such that both sides equal.

Let $H = H(\hat{\alpha}) = \mathcal{K}_3(G(\hat{\alpha}))$, i.e., H is the 3-graph whose hyperedges correspond to the triangles of $G(\hat{\alpha})$. It follows that the number of edges of H equals the number of triangles in G, i.e., for $d_{\hat{\alpha}} = \frac{3}{8}(1-\hat{\alpha})^3 + \frac{9}{4}(1-\hat{\alpha})^2\hat{\alpha}$

$$e(H) = (d_{\hat{\alpha}} + o(1))\binom{n}{3}.$$

On the other hand, every labeled copy of $K_{2,2,2}$ in G gives rise to a labeled $K^{(3)}_{2,2,2}$ in H, which gives rise to exactly one labeled Pasch-configuration (note, that in fact a copy of $K^{(3)}_{2,2,2}$ contains exactly two Pasch-configurations, however, those correspond to two different labelings of the same unlabeled copy of $K^{(3)}_{2,2,2}$). Moreover, every labeled copy of the Pasch-configuration P in H corresponds to a $K_{2,2,2}$ in G and, consequently,

$$N_P(H) = N_{K_{2,2,2}}(G) = (d_{\hat{\alpha}}^4 + o(1))n^6,$$

due to the choice of $\hat{\alpha}$. Obviously, $H = H(\hat{\alpha})$ is 5-partite and does not satisfy $\mathrm{DISC}_{d_{\hat{\alpha}}}$, which shows that it has the desired properties.

Moreover, we remark that the graph $G = G(\hat{\alpha})$ from above has the properties

$$N_{K_3}(G) = (d_{\hat{\alpha}} + o(1))n^3 \quad \text{and} \quad N_{K_{2,2,2}}(G) = (d_{\hat{\alpha}}^4 + o(1))n^6$$

while it obviously fails to satisfy $\mathrm{DISC}_{d_{\hat{\alpha}}}$ for graphs. This answers a question of Shapira and Yuster from [SYa].

3.6.6 Further concepts for uniform hypergraphs

In [Chu90] two further notions of quasi-randomness for k-uniform hypergraphs of density approximately $1/2$ were studied. The first one is disc_i:

disc_i For a k-uniform hypergraph H on n vertices define

$$\mathrm{disc}_i(H) := \frac{k!}{n^k} \max_{G \in \mathcal{H}_n^{(i-1)}} \Big| |\mathcal{K}_k(G) \cap E(H)| - |\mathcal{K}_k(G) \cap ([n]^k \setminus E(H))| \Big|,$$

where the maximum is taken over all $(i-1)$-uniform hypergraphs on n vertices,

and the other one is the property dev_i:

dev_i For a k-uniform hypergraph H on n vertices let

$$\mathrm{dev}_i(H) := \frac{2^{2^i}}{n^{k+i}} \sum_{\substack{u_\ell^1, u_\ell^2 \in V \\ 1 \le \ell \le i}} \sum_{\substack{u_\ell \in V \\ i < \ell \le k}} \prod_{\substack{t_j \in [2] \\ 1 \le j \le i}} (H(u_1^{t_1}, \dots, u_i^{t_i}, u_{i+1}, \dots, u_k) - 1/2),$$

where $H(f) = 1$ if f is a hyperedge of H and 0 otherwise.

In [Chu90] it is claimed that dev_i and disc_i are equivalent for all $i \in [k]$ (in the sense that if disc_i is small then dev_i is also small and vice versa). This is certainly the case for $i = k$ as it corresponds to the strong (octahedral) quasi-randomness, see also Section 3.6.1. However, the implication "if disc_i is small then dev_i is also small" is false [Chu10].

Below we give an example of a 3-uniform hypergraph H of density $1/2$ which has $\mathrm{disc}_2(H) = o(1)$, but fails to have $\mathrm{dev}_2(H) = o(1)$. More precisely, we give an example of a sequence of n-vertex 3-uniform hypergraphs H_n with $\mathrm{disc}_2(H_n) = o(1)$ and $\mathrm{dev}_2(H_n) > 3.3$, which contradicts assertion (ii) of Theorem 1 in the paper of Chung [Chu90].

The example is as follows. Consider a random graph $G(n, p)$ and let the edges of $H_{n,p}$ consist of those triplets of vertices of $G(n, p)$, which form a triangle in $G(n, p)$. Somewhat tedious calculations show that the expected number of (labeled) subgraphs of $K^{(3)}_{2,2,1}$ with precisely one or three hyperedges in $H_{n,p}$ is given by

$$(4p^3 - 8p^5 - 4p^6 + 16p^7 - 8p^8 + o(1))n^5\,.$$

For $p = (1/2)^{1/3}$ this gives approximately $0.395n^5$, which shows that in this case the expected numbers of odd and even subhypergraphs of $K^{(3)}_{2,2,1}$ deviate by at least $0.2n^5$, thus $\mathrm{dev}_2(H_n) \geq 3.3$ w.h.p. Standard arguments from random graph theory show that this holds w.h.p. for $H_{n,p}$ and due to the uniform distribution of the triangles of $G(n, p)$ for $p = (1/2)^{1/3}$ w.h.p. the hypergraph $H_{n,p}$ has density $1/2+o(1)$ and $\mathrm{disc}_2(H_{n,p}) = o(1)$. We believe the same example (letting H consist of those k-tuples which form $(\ell - 1)$-uniform cliques on k vertices in the random $(\ell-1)$-uniform hypergraph $G^{(\ell-1)}(n, p)$ with $p = (1/2)^{1/\binom{k}{\ell-1}}$) gives rise to a similar counterexample for $\mathrm{disc}_\ell \Rightarrow \mathrm{dev}_\ell$ for all ℓ and k with $2 \leq \ell < k$.

4 Fano-free hypergraphs

In this Chapter we prove that almost all hypergraphs without Fano planes are bipartite (see Theorem 1.7). Building on the ideas of this proof, we design an algorithm that colors every Fano-free 3-uniform hypergraph (and thus every bipartite 3-uniform hypergraph) properly in polynomial expected time, see Theorem 1.8.

First we collect some further notation and tools needed.

4.1 Further notation and tools

4.1.1 Definitions and notations

Here we study monotone properties of the type $\mathrm{Forb}(n, L)$ for a fixed hypergraph L, i.e., the family of all labeled hypergraphs on n vertices, which contain no copy of L as a (not necessarily induced) subgraph. The hypergraph we are interested in is the Fano plane. Recall, it is the unique triple system with 7 hyperedges on 7 vertices where every pair of vertices is contained in precisely one hyperedge (alternatively, one defines this hypergraph by identifying the points and the lines of the smallest projective plane – Fano plane – with vertices and hyperedges, respectively). The hypergraph of the Fano plane F is not 2-colorable, i.e., for every vertex partition $X \dot{\cup} Y = V(F)$ into two classes there exists an edge of F which is either contained in X or in Y. Consequently, $\mathrm{Forb}(n, F)$ contains all labeled bipartite 3-uniform hypergraphs on n vertices and we denote this set by $\mathcal{B}_n$.

It was shown independently by Füredi and Simonovits [FS05] and Keevash and Sudakov [KS05b], that the unique extremal Fano-free hypergraph for large n is the balanced, complete, bipartite hypergraph $B_n = (U \dot{\cup} W, E_{B_n})$, where $|U| = \lfloor n/2 \rfloor$, $|W| = \lceil n/2 \rceil$ and E_{B_n} consists of all hyperedges with at least one vertex in U and one vertex in W. Therefore, for the hypergraph of the Fano plane F we have

$$\mathrm{ex}(n, F) \;=\; e(B_n) \;=\; \binom{n}{3} - \binom{\lceil n/2 \rceil}{3} - \binom{\lfloor n/2 \rfloor}{3} \;=\; \frac{n^3}{8} - \frac{n^2}{4} - O(n) \;\leq\; \frac{n^3}{8} \tag{4.1}$$

and

$$\frac{3}{8}n^2 \geq \delta(B_n) = \left(\left\lceil \frac{n}{2} \right\rceil - 1 \right) \left\lfloor \frac{n}{2} \right\rfloor + \binom{\lfloor n/2 \rfloor}{2} \geq \frac{3}{8}n^2 - n\,. \tag{4.2}$$

Furthermore, we have

$$e(B_n) = e(B_{n-3}) + \delta(B_n) + \delta(B_{n-1}) + \delta(B_{n-2}).$$

In this chapter we consider only 3-uniform hypergraphs and by a hypergraph we will always mean a 3-uniform hypergraph. For the sake of a simpler notation we set

$$\mathcal{F}_n = \mathrm{Forb}(n, F)\,,$$

where in this chapter we always denote by F the hypergraph of the Fano plane. We will refer to hypergraphs not containing a copy of F as Fano-free hypergraphs.

We will use the following estimates (often without mentioning them explicitly). First of all we note that we can bound $|\mathcal{B}_n|$ by

$$2^{e(B_n)} \leq |\mathcal{B}_n| \leq 2^n \cdot 2^{e(B_n)}, \tag{4.3}$$

as there are at most 2^n partitions of $[n]$ in two disjoint sets and there are at most $e(B_n)$ hyperedges running between those two sets.

For $A \subseteq V(H)$ and $v \in V(H)$ denote by

$$L_A(v) = L_H(x)[A] = (A \setminus \{v\}, E_v \cap [A]^2)$$

the link of v induced on A.

Furthermore, we will use that for $n > 3k$ we have $\sum_{j<k} \binom{n}{j} < \binom{n}{k}$. Often we will omit floors and ceilings, as they will have no effect on our asymptotic arguments.

4.1.2 Tools

The following stability result for Fano-free hypergraphs was proved by Keevash and Sudakov [KS05b] and Füredi and Simonovits [FS05].

Theorem 4.1 (Stability theorem for Fano-free hypergraphs)**.** *For all $\alpha > 0$ there exists $\lambda > 0$[1] such that for every Fano-free hypergraph H on n vertices with at least $(\frac{1}{8} - \lambda)n^3$ hyperedges there exists a partition $V(H) = X \dot\cup Y$ so that $e(X) + e(Y) < \alpha n^3$.*

Recalling our Definition 1.10, Theorem 4.1 asserts that the Fano hypergraph F is 1-stable. We will use Theorem 2.7, the weak hypergraph regularity lemma for 3-uniform hypergraphs, and for the second algorithmic part we will need an algorithmic version of it, due to Czygrinow and Rödl [CR00], stated below.

Theorem 4.2 (Algorithmic weak regularity lemma)**.** *For every integer $t_0 \geq 1$ and every $\varepsilon > 0$, there exist $T_0 = T_0(t_0, \varepsilon)$, $n_0 = n_0(t_0, \varepsilon)$ and an algorithm* Regularize(H, ε, t_0)*, which for every 3-uniform hypergraph $H = (V, E)$ on $n \geq n_0$ vertices finds in $O(n^5 \log^2 n)$ time an ε-regular partition $V = V_1 \dot\cup \ldots \dot\cup V_t$ with $t_0 \leq t \leq T_0$.*

Remark 4.3. Actually, we do not need to know what triples in an ε-regular partition guaranteed by Regularize are ε-regular. As for our needs, we can simply try out all possible $(1-\varepsilon)\binom{t}{3}$ triples.

[1] It follows from the work in [FS05, KS05b] that $\lambda = \lambda(\alpha)$ is indeed a computable function.

In [KNRS10] a counting lemma, Lemma 2.6, for linear hypergraphs in the context of the weak regularity lemma was proved. The Fano plane F is linear, and thus Lemma 2.6 is applicable for F. Below we give (essentially) the same proof that first appeared in [KNRS10] under slightly relaxed conditions. First we will need some more definitions.

Let L be a hypergraph on the vertex set $[\ell]$ and let H be an ℓ-partite hypergraph with vertex partition $V(H) = V_1 \dot\cup \ldots \dot\cup V_\ell$. A copy L' of L in H, on the vertices $v_1 \in V_1, \ldots, v_\ell \in V_\ell$, is said to be *partite-isomorphic* to L if $i \mapsto v_i$ defines a hypergraph homomorphism.

For the lower bound of the counting lemma it is sufficient to know that the involved triples are "dense enough" on every "small subset", instead of being (ε, d)-regular. More precisely, we say a triple (V_1, V_2, V_3) of pairwise disjoint subsets $V_1, V_2, V_3 \subseteq V$ is *one-sided* (ε, d)*-regular* for $\varepsilon > 0$ and $d \geq 0$ if

$$d_H(W_1, W_2, W_3) \geq d$$

for all triples of subsets $W_1 \subseteq V_1, W_2 \subseteq V_2, W_3 \subseteq V_3$ satisfying $|W_i| \geq \varepsilon |V_i|$, $i = 1, 2, 3$. Note also, that an ε-regular triple of density d is one-sided $(\varepsilon, d - \varepsilon)$-regular.

Theorem 4.4 (Key-lemma). *For every $\ell \in \mathbb{N}$ and $d > 0$ there exist $\varepsilon = \varepsilon(\ell, d) > 0$ and a positive integer $m_0 = m_0(\ell, d)$ with the following property.*

If H is an ℓ-partite 3-uniform hypergraph with vertex classes $V_1, \ldots, V_\ell$, such that $|V_1| = \ldots = |V_\ell| \geq m_0$, and L is a linear hypergraph on ℓ vertices such that for every $e \in E(L)$ the triple $(V_i)_{i \in e}$ is one-sided (ε, d)-regular. Then H contains a copy of L.

Proof. We will prove the following stronger assertion, which gives a lower bound on the number of partite-isomorphic copies of L in H.

Proposition 4.5. *For every $\ell \in \mathbb{N}$ and γ, $d > 0$, there exist $\varepsilon = \varepsilon(\ell, \gamma, d) > 0$ and $m_0 = m_0(\ell, \gamma, d)$ so that the following holds.*

Let $L = ([\ell], E(L))$ be a linear hypergraph and let $H = (V_1 \dot\cup \ldots \dot\cup V_\ell, E)$ be an ℓ-partite, 3-uniform hypergraph where $|V_1| = \cdots = |V_\ell| \geq m_0$. If for all edges $e = \{i, j, k\} \in E(L)$, the triple (V_i, V_j, V_k) is one-sided (ε, d_e)-regular for some $d_e \geq d$, then

$$N_L(V_1, \ldots, V_\ell) \geq (1 - \gamma) \prod_{e \in E(L)} d_e \prod_{i \in [\ell]} |V_i| \,.$$

Let $\ell \in \mathbb{N}$ and $\gamma, d > 0$ be fixed. We shall prove, by induction on $|E(L)|$, that $\varepsilon = \gamma (d/2)^{|E(L)|}$ will suffice to estimate the lower bound on copies of L, provided m_0 is large enough. If $|E(L)| = 0$ or $|E(L)| = 1$, the result is trivial. It is also easy to see that the result holds whenever L consists of pairwise disjoint hyperedges, since then the number of partite-isomorphic copies of L in H is at least $\prod_{e \in E(L)} d_e \prod_{i \in [\ell]} |V_i|$.

For the general case let m_0 be large enough, so that we can apply the induction assumption on $|E(L)| - 1$ edges with precision $\gamma/2$ and d (and note that $\varepsilon = \gamma (d/2)^{|E(L)|} \leq (\gamma/2)(d/2)^{|E(L)|-1}$). All copies of various subhypergraphs discussed below are tacitly assumed to be partite-isomorphic.

Let L have $|E(L)| \geq 2$ edges and let $H = (V, E)$ be a 3-uniform hypergraph satisfying the assumptions of Proposition 4.5. Fix an edge $e_0 \in E(L)$ and let $L_- = ([\ell], E(L) \setminus \{e_0\})$ be the hypergraph obtained from L by removing the edge e_0. Moreover, for a copy L'_- of L_- in H, we denote by $e_0(L'_-)$ the unique triple of vertices which together with L'_- forms a copy of L in H. Furthermore, let $\mathbf{1}_E \colon [V]^3 \to \{0, 1\}$ be the indicator function of the edge set E of H. With this notation, a copy L'_- of L_- in H extends to a copy of L if, and only if, $\mathbf{1}_E(e_0(L'_-)) = 1$. Consequently, summing over all copies L'_- of L_- in H, we obtain a formula on the number $|\{L \subseteq H\}|$ of copies of L in H:

$$\begin{aligned} |\{L \subseteq H\}| = \sum_{L'_- \subseteq H} \mathbf{1}_E(e_0(L'_-)) &= \sum_{L'_- \subseteq H} (d_{e_0} + \mathbf{1}_E(e_0(L'_-)) - d_{e_0}) \\ &= d_{e_0} |\{L_- \subseteq H\}| + \sum_{L'_- \subseteq H} (\mathbf{1}_E(e_0(L'_-)) - d_{e_0}). \end{aligned}$$

Using the induction assumption for L_- we infer

$$|\{L \subseteq H\}| \geq (1 - \tfrac{\gamma}{2}) \prod_{e \in E(L)} d_e \prod_{i \in [\ell]} |V_i| + \sum_{L'_- \subseteq H} (\mathbf{1}_E(e_0(L'_-)) - d_{e_0}). \tag{4.4}$$

We bound the error term $\sum_{L'_- \subseteq H} (\mathbf{1}_E(e_0(L'_-)) - d_{e_0})$ from below. For that, we will appeal to the one-sided regularity of $(V_i)_{i \in e_0}$. Let $L_* = L[[\ell] \setminus e_0]$ be the induced subhypergraph of L obtained by removing the vertices of e_0 and all edges of L intersecting e_0. For a copy L'_* of L_* in H, let $\operatorname{ext}(L'_*)$ be the set of triples $T \in \prod_{i \in e_0} V_i$ such that $V(L'_*) \dot\cup T$ spans a copy of L'_- in H. Hence,

$$\sum_{L'_- \subseteq H} (\mathbf{1}_E(e_0(L'_-)) - d_{e_0}) = \sum_{L'_* \subseteq H} \sum_{T \in \operatorname{ext}(L'_*)} (\mathbf{1}_E(T) - d_{e_0})$$

and, moreover, since L is a linear hypergraph, we have $|e_0 \cap e| \leq 1$ for every edge e of L_-. Thus, for every fixed copy L'_* of L_* in H and every $i \in e_0$, there exists a subset $W_i^{L'_*} \subseteq V_i$ such that

$$\operatorname{ext}(L'_*) = \prod_{i \in e_0} W_i^{L'_*}. \tag{4.5}$$

Indeed, for every $i \in e_0$, the set $W_i^{L'_*}$ consists of those vertices $v \in V_i$ with the property that $V(L'_*) \dot\cup \{v\}$ spans a copy of L induced on $V(L_*) \dot\cup \{i\}$ in H. Therefore, we can bound the error term as follows

$$\begin{aligned} \sum_{L'_- \subseteq H} (\mathbf{1}_E(e_0(L'_-)) - d_{e_0}) &= \sum_{L'_* \subseteq H} \sum \Big\{ \mathbf{1}_E(T) - d_{e_0} \colon T \in \prod_{i \in e_0} W_i^{L'_*} \Big\} \\ &\geq - \sum_{L'_* \subseteq H} \varepsilon \prod_{i \in e_0} |V_i| \geq -\frac{\gamma}{2} \prod_{e \in E(L)} d_e \prod_{i \in [\ell]} |V_i|, \end{aligned}$$

where the one-sided (ε, d_{e_0})-regularity, the choice of ε, and (4.5) were used for the last two estimates. Now the proposition follows from (4.4), which implies immediately Theorem 4.4. □

4.2 Almost all Fano-free hypergraphs are bipartite

4.2.1 Outline of the proof of Theorem 1.7

Definition 4.6. *With every hypergraph $H \in \mathcal{F}_n$ we associate a partition $X_H \dot{\cup} Y_H$ of its vertex set $V(H)$ that minimizes $e(X_H) + e(Y_H)$. In case of ambiguity fix one such partition arbitrarily. Furthermore, by X_H and Y_H, we mean this partition of the vertices of H.*

We employ the so-called Kleitman-Rothschild method, which is best explained with the help of a concrete example. Our proof will be split into several lemmas, and will follow similar steps as in [BBS04]. Namely, we will study several subclasses of $\mathcal{F}_n$ that for appropriately chosen parameters α and $\beta > 0$ form the chains

$$\mathcal{F}_n \supseteq \mathcal{F}'_n(\alpha) \supseteq \mathcal{F}''_n(\alpha,\beta) \supseteq \mathcal{F}'''_n(\alpha,\beta)$$

and

$$\mathcal{F}_n \supseteq \mathcal{B}_n \supseteq \mathcal{F}'''_n(\alpha,\beta).$$

Roughly speaking, we will show that

$$|\mathcal{F}'_n(\alpha)| \geq (1-o(1))|\mathcal{F}_n|\,,\ |\mathcal{F}''_n(\alpha,\beta)| \geq (1-o(1))|\mathcal{F}_n|$$
$$\text{and} \quad |\mathcal{F}'''_n(\alpha,\beta)| \geq (1-o(1))|\mathcal{F}_n|$$

and due to

$$\begin{aligned}|\mathcal{F}_n| \leq |\mathcal{F}_n \setminus \mathcal{F}'_n(\alpha)| &+ |\mathcal{F}'_n(\alpha) \setminus \mathcal{F}''_n(\alpha,\beta)| \\ &+ |\mathcal{F}''_n(\alpha,\beta) \setminus \mathcal{F}'''_n(\alpha,\beta)| + |\mathcal{B}_n|\end{aligned}$$

Theorem 1.7 then follows. Below we informally define all these special subclasses of Fano-free hypergraphs and sketch the main ideas of the proof.

1. $\mathcal{F}'_n(\alpha) \subseteq \mathcal{F}_n$ will be the class of "almost bipartite" hypergraphs, i.e., those Fano-free hypergraphs that admit a partition of its vertices into classes of nearly equal size, such that less than αn^3 edges lie inside the partition classes. Using the weak hypergraph regularity lemma (Theorem 2.7), the counting lemma (Lemma 2.6), and the stability theorem (Theorem 4.1), we will upper bound the number of hypergraphs that are not in $\mathcal{F}'_n(\alpha)$.

2. $\mathcal{F}''_n(\alpha,\beta)$ will denote the set of those hypergraphs that are "dense everywhere" in the sense that whenever we take three disjoint subsets of vertices, say W_1, W_2, W_3, not all of them contained in X_H or Y_H, the number of hyperedges that run between them will be at least $d|W_1||W_2||W_3|$ for some positive constant $d > 0$. The

proof of this fact is a straightforward counting argument. Moreover, we will also show that for every $H \in \mathcal{F}''_n(\alpha, \beta)$ the degrees of vertices inside their own partition class, that is X_H or Y_H, are "small".

3. The last class of hypergraphs will be $\mathcal{F}'''_n(\alpha, \beta)$. For members H of this class we demand that the joint link of every set of 3 vertices of any of the two partition classes X_H and Y_H must contain a K_4. Instead of proving $|\mathcal{F}'''_n(\alpha, \beta)| \geq (1 - o(1))|\mathcal{F}_n|$ directly, we will use this class of hypergraphs in order to estimate $\mathcal{F}_n$ inductively.

4.2.2 Almost bipartite hypergraphs.

Our first step for the proof of Theorem 1.7 is an estimate on the number of those hypergraphs $H \in \mathcal{F}_n$ which are far from being bipartite, namely for which $e(X_H) + e(Y_H) \geq \alpha n^3$ for some $\alpha > 0$ to be specified later. Thus, the remaining hypergraphs will admit a "nice" partition. Moreover, most of these remaining hypergraphs H will have partition classes of nearly same size.

Definition 4.7.

$$\mathcal{F}'_n(\alpha) = \{H \in \mathcal{F}_n\colon\ e(X_H) + e(Y_H) < \alpha n^3 \ \textit{and}\ |X_H|, |Y_H| < n/2 + 2\sqrt{h(6\alpha)}n\}\,.$$

Lemma 4.8. *For every $\alpha \in (0, \frac{1}{12})$ there exist $c' > 0$ and an integer n'_0 such that for all $n \geq n'_0$*

$$|\mathcal{F}_n \setminus \mathcal{F}'_n(\alpha)| < 2^{e(B_n) - c'n^3}\,.$$

Proof. The proof of Lemma 4.8 combines the weak hypergraph regularity lemma with the stability theorem for Fano-free hypergraphs applied to the cluster hypergraph.

Let $\lambda = \lambda(\alpha/2)$ be given by Theorem 4.1. We may assume $\lambda < 16h(6\alpha)$. We set

$$c' = \frac{\lambda}{17}\,.$$

We choose η such that $\lambda > 16h(6\eta)$ and $\eta \leq \alpha/2$. Finally let $\varepsilon = \varepsilon(\eta/2) \leq \eta/2$ be given by Lemma 2.6. Set $t_0 = 1/\varepsilon$ and let n be sufficiently large, in particular, set $n'_0 \gg \max\{T_0, n_0\}$, where T_0 and n_0 are given by the weak regularity lemma, Theorem 2.7. For the main steps of the proof it is sufficient to keep in mind that

$$0 < \varepsilon = t_0^{-1} \leq \eta \ll \lambda \ll \alpha.$$

We may assume in the following that t divides n, and thus $|V_i| = n/t, i = 1, \ldots, t$, as this does not affect our asymptotic considerations.

We will upper bound $|\mathcal{F}_n \setminus \mathcal{F}'_n(\alpha)|$ in two steps. In the first step we bound the number of hypergraphs H that have $e(X_H) + e(Y_H) \geq \alpha n^3$. In the second step we show that most of the hypergraphs H with $e(X_H) + e(Y_H) < \alpha n^3$ will have nearly equal sizes:

$$\max\{|X_H|, |Y_H|\} < \frac{n}{2} + 2\sqrt{h(6\alpha)}n\,.$$

Step 1.

Consider a hypergraph $H \in \mathcal{F}_n$ satisfying $e(X_H) + e(Y_H) \geq \alpha n^3$. We apply the weak regularity lemma, Theorem 2.7, with parameters ε and t_0. Firstly, we estimate the number of hyperedges, which are contained in the "uncontrolled" part of the regular partition:

- the number of hyperedges intersecting at most two of the clusters is at most
$$t\binom{n/t}{2}n < \frac{1}{2t}n^3,$$
- the number of hyperedges contained in irregular triples is at most
$$\varepsilon\binom{t}{3}\left(\frac{n}{t}\right)^3 < \frac{\varepsilon}{6}n^3,$$
- the number of hyperedges that are contained in ε-regular triples of density less than η is at most
$$\eta\left(\frac{n}{t}\right)^3\binom{t}{3} < \frac{\eta}{6}n^3.$$

Thus, the number of discarded edges is less than ηn^3.

Secondly, consider the resulting cluster-hypergraph $H(\eta)$. It must be Fano-free as otherwise Lemma 2.6 would imply that H also contains a copy of the hypergraph of the Fano plane. We assumed that $e(X_H) + e(Y_H) \geq \alpha n^3$, so we can bound the number of hyperedges in $H(\eta)$ from above by $(1-\lambda)t^3/8$. Otherwise, Theorem 4.1 would give us a partition of $V_1, \dots, V_t$ into disjoint sets X and Y with $e_{H(\eta)}(X) + e_{H(\eta)}(Y) < \alpha t^3/2$. Defining a partition of $V(H)$ into the following two sets

$$A = \bigcup_{U \in X} U \quad \text{and} \quad B = \bigcup_{W \in Y} W,$$

with

$$e_H(A) + e_H(B) < \eta n^3 + \frac{\alpha}{2}t^3\left(\frac{n}{t}\right)^3 \leq \alpha n^3\,,$$

which yields a contradiction to $e(X_H) + e(Y_H) \geq \alpha n^3$.

Now we are able to bound the number of hypergraphs $H \in \mathcal{F}_n$ with $e(X_H) + e(Y_H) \geq \alpha n^3$ from above by calculating the total possible number of ε-regular partitions together with all possible cluster-hypergraphs associated with them and all possible hypergraphs that could give rise to such a particular cluster-hypergraph. This way we get

$$|\mathcal{F}_n \setminus \{H \in \mathcal{F}_n \colon\ e(X_H) + e(Y_H) \geq \alpha n^3\}| \leq \sum_{t=t_0}^{T_0} t^n \cdot 2^{\binom{t}{3}} \cdot 2^{(1-\lambda)\frac{t^3}{8}(\frac{n}{t})^3} \cdot \left(\sum_{j=0}^{\eta n^3 - 1} \binom{\binom{n}{3}}{j}\right)$$
$$\leq T_0^{n+1} \cdot 2^{\binom{T_0}{3}} \cdot 2^{(1-\lambda)n^3/8} \cdot \binom{\binom{n}{3}}{\eta n^3} \leq 2^{(n+1)\log T_0 + \binom{T_0}{3} + n^3/8 - \lambda n^3/8 + h(6\eta)n^3/6}$$
$$< 2^{n^3/8 - \lambda n^3/16},$$

for sufficiently large n, due to the choice of λ.

Step 2.

We now estimate the number of those hypergraphs H, for which $e(X_H) + e(Y_H) < \alpha n^3$, but $\max\{|X_H|, |Y_H|\} \geq n/2 + 2\sqrt{h(6\alpha)}n$. First we upper bound $e(X_H, Y_H)$ for such a hypergraph H by

$$e(X_H, Y_H) \leq |X_H|\binom{|Y_H|}{2} + |Y_H|\binom{|X_H|}{2} < \frac{n}{2}|X_H||Y_H| < \frac{n^3}{8} - 2h(6\alpha)n^3.$$

Note that there are at most 2^n possible partitions, and since less than αn^3 hyperedges are completely contained in X_H and Y_H, those hyperedges can be chosen in at most

$$\sum_{i=0}^{\alpha n^3 - 1} \binom{\binom{n}{3}}{i} \leq \binom{\binom{n}{3}}{\alpha n^3}$$

ways. Finally, as we assumed that our partitions are "unbalanced" we estimate the number of possible choices of hyperedges between X_H and Y_H by $2^{n^3/8 - 2h(6\alpha)n^3}$. Altogether we get, that there are at most

$$2^n \cdot \binom{\binom{n}{3}}{\alpha n^3} \cdot 2^{n^3/8 - 2h(6\alpha)n^3} \leq 2^{n + h(6\alpha)n^3/6 + n^3/8 - 2h(6\alpha)n^3} \leq 2^{n^3/8 - h(6\alpha)n^3}$$

hypergraphs with $e(X_H) + e(Y_H) < \alpha n^3$ and

$$\max\{|X_H|, |Y_H|\} \geq \frac{n}{2} + 2\sqrt{h(6\alpha)}n.$$

Combining Step 1 and 2 we obtain

$$|\mathcal{F}_n \setminus \mathcal{F}'_n(\alpha)| \leq 2^{n^3/8 - \lambda n^3/16} + 2^{n^3/8 - h(6\alpha)n^3} < 2^{n^3/8 - \lambda n^3/16 + 1},$$

since $h(6\alpha) > \lambda/16$. Due to $n^3/8 - e(B_n) \leq n^2/4 + O(n)$ and the choice of $c' = \lambda/17$, the lemma follows for sufficiently large n. $\square$

4.2.3 Everywhere dense hypergraphs.

Now we know that almost all Fano-free hypergraphs are nearly bipartite and admit a partition into almost equal classes. We want to restrict our consideration to those hypergraphs that in addition have no sparse "bipartite" spots. Our motivation comes from random bipartite hypergraphs. Namely, we would expect $\frac{1}{2}N\binom{N}{2}$ edges having exactly one end in the first class and two in the second in $\mathcal{H}(N, N, 1/2)$, the random bipartite hypergraph with both classes of size N where each edge exists with probability $1/2$. If we would take any three disjoint subsets not all of them in one partition class, each of size, say m, then we would expect there $m^3/2$ hyperedges. Deviations from this value, say only $m^3/4$ edges instead of $m^3/2$, would only happen with very small probability. The following lemma, Lemma 4.10, states that this intuition holds for almost all hypergraphs in $\mathcal{F}'_n(\alpha)$.

Definition 4.9. *Let $\mathcal{F}''_n(\alpha, \beta)$ denote the family of those hypergraphs $H \in \mathcal{F}'_n(\alpha)$, for which the following condition holds.*

For any pairwise disjoint sets $W_1 \subset X_H$, $W_2 \subset Y_H$ and $W_3 \subset Z_H$, where $Z_H \in \{X_H, Y_H\}$, with $|W_i| \geq \beta n$ for $i = 1, 2, 3$ we have

$$e_H(W_1, W_2, W_3) \geq \frac{1}{4}|W_1||W_2||W_3|.$$

The following lemma shows that most hypergraphs in $\mathcal{F}'_n(\alpha)$ belong to $\mathcal{F}''_n(\alpha, \beta)$.

Lemma 4.10. *For every $\beta > 0$ there exist α, $c'' > 0$ and an integer n''_0 such that for all $n \geq n''_0$*

$$|\mathcal{F}'_n(\alpha) \setminus \mathcal{F}''_n(\alpha, \beta)| < 2^{e(B_n) - c'' n^3}.$$

Proof. Choose $\alpha > 0$ such that

$$\beta^3 \left(1 - h\left(1/4\right)\right) \geq h(6\alpha)/3\,,$$

set $c'' = h(6\alpha)/7$, and let n''_0 be sufficiently large.

Below we bound the number of hypergraphs H with $H \in \mathcal{F}'_n(\alpha) \setminus \mathcal{F}''_n(\alpha, \beta)$. There are at most 2^n partitions $X_H \dot\cup Y_H = [n]$ of the vertex set and we can choose the edges lying completely within X_H and Y_H in at most

$$\sum_{j=0}^{\alpha n^3 - 1} \binom{\binom{n}{3}}{j} \leq \binom{\binom{n}{3}}{\alpha n^3}$$

possible ways. A simple averaging argument shows that it suffices to consider sets W_i with $|W_i| = \beta n$ and there are at most

$$2\binom{n/2 + 2\sqrt{h(6\alpha)}n}{\beta n}^3 \sum_{0 \leq i < \beta^3 n^3/4} \binom{\beta^3 n^3}{i} < 2^{3n+1} \binom{\beta^3 n^3}{\beta^3 n^3/4}$$

ways to select W_1, W_2, W_3 and the hyperedges in $e_H(W_1, W_2, W_3)$. Finally, there are at most

$$2^{e(B_n)-\beta^3 n^3}$$

ways to choose the remaining edges of H. Multiplying everything together, we obtain

$$\begin{aligned}|\mathcal{F}'_n(\alpha) \setminus \mathcal{F}''_n(\alpha,\beta)| &\le 2^{4n+1}\binom{\binom{n}{3}}{\alpha n^3}\binom{\beta^3 n^3}{\beta^3 n^3/4}2^{e(B_n)-\beta^3 n^3}\\ &\le 2^{4n+1+h(6\alpha)n^3/6+h(1/4)\beta^3 n^3+e(B_n)-\beta^3 n^3} \le 2^{e(B_n)-c'' n^3},\end{aligned}$$

for sufficiently large n. □

We will also need the following useful observation, that for suitably chosen α and β, every $H \in \mathcal{F}''_n(\alpha,\beta)$ has no vertex of high degree in its own partition class.

Lemma 4.11. *For every $\gamma > 0$ there exist α, $\beta > 0$ and an integer n_0, such that for every $H \in \mathcal{F}''_n(\alpha,\beta)$ we have*

$$\max\{\Delta(H[X_H]), \Delta(H[Y_H])\} < \gamma n^2,$$

for all $n \ge n_0$.

For the proof of Lemma 4.11, we will use a simple consequence of the regularity lemma for graphs, Theorem 2.2 (for a better dependency of the constants one also could also use [PRR02, Theorem 1.1]).

Theorem 4.12. *For every $\gamma > 0$ and $\varepsilon \in (0, \gamma/3)$ there exist T_0, N_0 such that the following holds.*

For all vertex disjoint graphs G_X and G_Y on $|V(G_X)| + |V(G_Y)| = n \ge N_0$ vertices with $e(G_X)$, $e(G_Y) \ge \gamma n^2$ there exist $t \le T_0$ and pairwise disjoint sets $X_1, X_2, Y_1, Y_2, Y_3, Y_4$, each of size n/t, and $X_1, X_2 \subset V(G_X)$ and $Y_i \subset V(G_Y)$, $i \in [4]$, so that $G_X[X_1, X_2]$, $G_Y[Y_1, Y_2]$ and $G_Y[Y_3, Y_4]$ are ε-regular with density at least $\gamma/3$.

With this result at hand we can give the proof of Lemma 4.11.

Proof. Let $\varepsilon = \min\{\frac{1}{2}\varepsilon(\gamma/6), \gamma/6\}$, where $\varepsilon(\gamma/6)$ is given by the key lemma, Theorem 4.4. Set $\beta = \varepsilon/(2T_0)$, with $T_0 = T_0(\gamma, \varepsilon)$ given by Theorem 4.12. Let $\alpha = \alpha(\beta)$ be given by Lemma 4.10 and let n_0 be sufficiently large. Again, it is sufficient to keep in mind:

$$0 \ll \alpha \ll \beta \ll T_0^{-1} \ll \varepsilon \ll \gamma.$$

We prove our lemma by contradiction. More precisely, we will assume that there exists a hypergraph $H \in \mathcal{F}''_n(\alpha,\beta)$ with $\max\{\Delta(H[X_H]), \Delta(H[Y_H])\} \ge \gamma n^2$, and we will show that H contains a copy of the hypergraph of the Fano plane.

Without loss of generality assume that there exists $H \in \mathcal{F}''_n(\alpha,\beta)$ and a vertex $x \in X_H$ with $\deg_{H[X]}(x) \ge \gamma n^2$. Thus, $e(L_Y(x)) \ge e(L_X(x)) \ge \gamma n^2$, as otherwise this violates the minimality condition of the partition $X_H \dot{\cup} Y_H = V(H)$. We consider the graphs

$$G_X = L_X(x) = (X_H \setminus \{x\}, E_x \cap \tbinom{X}{2}) \quad \text{and} \quad G_Y = L_Y(x) = (Y_H, E_x \cap \tbinom{Y}{2})$$

and apply Theorem 4.12 to $G_X \dot{\cup} G_Y$. This way we obtain ε-regular pairs $(X_1, X_2) \subset G_X$ and $(Y_1, Y_2), (Y_3, Y_4) \subset G_Y$, with $|X_i| = |Y_j| \geq (n-1)/T_0$ and $i \in [2]$, $j \in [4]$, each of density at least $\gamma/3$.

Consider the following 7-partite subhypergraph L with vertex classes $\{x\}$, X_1, X_2, Y_1, Y_2, Y_3, and Y_4. Denote L^x to be the hypergraph obtained from L by blowing up its first vertex class $\{x\}$ to the size of X_1 (all other partition classes are equal), and denote this blown-up class by $\widetilde{X}$. More precisely, $L^x = (W^x, E^x)$, where

$$W^x = \widetilde{X} \dot{\cup} X_1 \dot{\cup} X_2 \dot{\cup} Y_1 \dot{\cup} Y_2 \dot{\cup} Y_3 \dot{\cup} Y_4$$

and

$$\{a,b,c\} \in E^x \quad \Longleftrightarrow \quad \begin{cases} \{a,b,c\} \in E(L)\,, & \text{if } \{a,b,c\} \cap \widetilde{X} = \varnothing\,, \\ \{x,b,c\} \in E(L)\,, & \text{if } a \in \widetilde{X} \text{ and } b,\, c \notin \widetilde{X}\,. \end{cases}$$

Note, that L contains a copy of the hypergraph of the Fano plane if, and only if, L^x contains one. Now we apply Theorem 4.4 to L^x, as L^x contains now 7 one-sided $(\varepsilon, \gamma/6)$-regular triples and these triples form a Fano plane. This is true since the triples $(\widetilde{X}, X_1, X_2)$, $(\widetilde{X}, Y_1, Y_2)$ and $(\widetilde{X}, Y_3, Y_4)$ "inherit" the ε-regularity from the ε-regular pairs of (X_1, X_2), (Y_1, Y_2), and (Y_3, Y_4), while the other triples are one-sided $(\varepsilon, \gamma/6)$-regular due to the choice of β and the properties of $H \in \mathcal{F}''_n(\alpha, \beta)$. This yields a contradiction and Lemma 4.11 follows. □

4.2.4 Proof of Theorem 1.7.

We will need the following consequence from Janson's inequality [Jan90], Theorem 2.20.

Lemma 4.13. *The probability that the binomial random graph $\mathcal{G}(m, \frac{1}{8})$ with $m \geq 253$ vertices and edge probability $1/8$ does not contain a copy of K_4 is bounded from above by* $\exp(-2^{-11}m^2)$.

Proof. Let $t_1, \ldots, t_{\binom{m}{2}}$ be jointly independent Boolean random variables representing the edges of $\mathcal{G}(m, \frac{1}{8})$. Let the collection of those 6-sets of the set $\binom{[m]}{2}$ that correspond to K_4's be denoted by $\mathcal{A}$. Therefore, the random variable $X = \sum_{A \in \mathcal{A}} \prod_{j \in A} t_j$ counts the number of K_4's in $\mathcal{G}(m, \frac{1}{8})$. Applying Janson's inequality [Jan90], Theorem 2.20, we bound the probability of the event $X = 0$ by

$$\Pr(X = 0) \leq \exp\left(-\frac{\mathbb{E}(X)^2}{2\Delta}\right),$$

where $\mathbb{E}(X) = (\frac{1}{8})^6\binom{m}{4}$ is the expectation and

$$\Delta = \sum_{A,B\in\mathcal{A}:A\cap B\neq\varnothing} \mathbb{E}\left(\prod_{j\in A\cup B} t_j\right) \leq \mathbb{E}(X) \sup_{A\in\mathcal{A}} \sum_{B\in\mathcal{A}:A\cap B\neq\varnothing} \mathbb{E}\left(\prod_{j\in B\setminus A} t_j\right)$$
$$\leq \mathbb{E}(X)\left(\binom{m}{2}\cdot 6\cdot\left(\frac{1}{8}\right)^5 + m\cdot 4\cdot\left(\frac{1}{8}\right)^3\right) \leq \mathbb{E}(X)\left(\frac{3m^2+253m}{2^{15}}\right) \leq \mathbb{E}(X)\frac{m^2}{2^{13}}.$$

Hence, we obtain

$$\Pr(X=0) \leq \exp\left(-\frac{2^{12}\mathbb{E}(X)}{m^2}\right) = \exp\left(-\frac{\binom{m}{4}}{2^6m^2}\right) \leq \exp\left(-2^{-11}m^2\right).$$

□

We finally define the last subclass of Fano-free hypergraphs.

Definition 4.14. *Let $\mathcal{F}_n'''(\alpha,\beta)$ denote the family of those hypergraphs $H \in \mathcal{F}_n''(\alpha,\beta)$, for which the following condition holds.*

For all triples $z_1, z_2, z_3 \in Z$ of vertices with $Z \in \{X_H, Y_H\}$ we have $L_Q(z_1)\cap L_Q(z_2)\cap L_Q(z_3) \supseteq K_4$, where $\{Q,Z\} = \{X_H, Y_H\}$. In other words, we require that the common link of any triple from X_H or Y_H contains a copy of K_4 in the other vertex class.

It follows directly from the definition, that every $H \in \mathcal{F}_n'''(\alpha,\beta)$ is bipartite, i.e., $\mathcal{F}_n'''(\alpha,\beta) \subseteq \mathcal{B}_n$. Otherwise, any hyperedge e, say in X_H, together with the K_4 in Y_H, which lies in the common link of the vertices of e would span a copy of the hypergraph of the Fano plane. We also note that we could replace K_4 in the definition of $\mathcal{F}_n'''(\alpha,\beta)$ by a 1-factor of K_4 that is created by the union of the links of any three vertices.

We are now going to prove our main theorem, Theorem 1.7, by induction. The proof is based on the lemmas from the previous sections.

Proof of Theorem 1.7. We set

$$\vartheta = 2^{-17} \quad \text{and} \quad c = \frac{\vartheta}{3} \tag{4.6}$$

and choose $\gamma > 0$ such that

$$3h(2\gamma) < \vartheta\,. \tag{4.7}$$

Let α and $\beta > 0$ be given by Lemma 4.11. We may also assume that

$$3\sqrt{h(6\alpha)} + 6h(6\alpha) < \vartheta/2\,, \tag{4.8}$$

as choosing α smaller we will only have to eventually increase n_0. Again, it is sufficient to keep in mind that

$$0 < \alpha \ll \beta \ll \gamma \ll \vartheta = 2^{-17}\,.$$

Let c' and c'' be given by Lemma 4.8 and Lemma 4.10. Finally, let

$$n_0 \geq \max\{2^{20}, 14/\vartheta, 1/c', 1/c''\}$$

be sufficiently large so that Lemma 4.8, Lemma 4.10 and Lemma 4.11 hold.

By induction on n we will verify the following statement, which implies Theorem 1.7

$$|\mathcal{F}_n| \leq |\mathcal{B}_n|(1 + 2^{n_0^2 n - cn^2}) . \tag{4.9}$$

For $n \leq n_0$ the statement is trivial, since then $2^{n_0^2 n - cn^2}$ is bigger than the number of all hypergraphs on n vertices and we now proceed with the induction step and verify (4.9) for $n > n_0$.

The proof is based on the following chains

$$\mathcal{F}_n \supseteq \mathcal{F}'_n(\alpha) \supseteq \mathcal{F}''_n(\alpha, \beta) \supseteq \mathcal{F}'''_n(\alpha, \beta),$$

and

$$\mathcal{F}_n \supseteq \mathcal{B}_n \supseteq \mathcal{F}'''_n(\alpha, \beta).$$

Consequently,

$$|\mathcal{F}_n| \leq |\mathcal{F}_n \setminus \mathcal{F}'_n(\alpha)| + |\mathcal{F}'_n(\alpha) \setminus \mathcal{F}''_n(\alpha, \beta)| + |\mathcal{F}''_n(\alpha, \beta) \setminus \mathcal{F}'''_n(\alpha, \beta)| + |\mathcal{B}_n| .$$

Lemma 4.8 bounds $|\mathcal{F}_n \setminus \mathcal{F}'_n(\alpha)|$ and Lemma 4.10 bounds $|\mathcal{F}'_n(\alpha) \setminus \mathcal{F}''_n(\alpha, \beta)|$. Hence, it remains to estimate $|\mathcal{F}''_n(\alpha, \beta) \setminus \mathcal{F}'''_n(\alpha, \beta)|$.

For that we will use the induction assumption and proceed as follows. Let $H \in \mathcal{F}''_n(\alpha, \beta) \setminus \mathcal{F}'''_n(\alpha, \beta)$ and $X_H \dot{\cup} Y_H$ be its minimal partition. Consider a subset $S \in \binom{X_H}{3} \dot{\cup} \binom{Y_H}{3}$. Deleting S from H, we obtain a Fano-free hypergraph H' on $n-3$ vertices, where $V(H') = [n] \setminus S$. On the other hand, for every $H \in \mathcal{F}''_n(\alpha, \beta) \setminus \mathcal{F}'''_n(\alpha, \beta)$ there exists a hypergraph $H' \in \mathcal{F}_{n-3}$ such that H can be reconstructed from H' in the following way. For $H' \in \mathcal{F}_{n-3}$ we choose a set S of 3 vertices, which we "connect" in an appropriate manner, so that the resulting hypergraph is in $\mathcal{F}''_n(\alpha, \beta) \setminus \mathcal{F}'''_n(\alpha, \beta)$.

We can choose the set S, the partition of H' and the set which contains S in at most

$$\binom{n}{3} 2^{n-3}$$

ways. Since $H \in \mathcal{F}''_n(\alpha, \beta)$ and Lemma 4.11 holds, we also know that every vertex in S has at most γn^2 neighbors in its own partition class. This again bounds the number of ways for choosing these hyperedges by

$$\left(\sum_{j=0}^{\gamma n^2 - 1} \binom{\binom{n}{2}}{j} \right)^3 \leq \binom{\binom{n}{2}}{\gamma n^2}^3 .$$

For every vertex in S we have at most $2^{n^2/4}$ possibilities for choosing edges with one more end in the same partition as S and the other end in the other partition class, this gives us at most

$$2^{3n^2/4}$$

ways to choose that type of hyperedges. The last estimate concerns the number of ways we can connect our triple S to the other partition class, say Y, without creating any single copy of K_4, which is contained in the joint link of the vertices from S. Here we use Lemma 4.13. For every vertex v in S, say $S \subset X$, we can choose its link graph $L_Y(v)$ in at most $2^{\binom{|Y|}{2}}$ ways. However, since the joint link of three vertices in S contains no K_4, we infer from Lemma 4.13, that there are at most

$$2^{3\binom{|Y|}{2}} \exp(-2^{-11}|Y|^2) < 2^{3\binom{|Y|}{2}-|Y|^2/2^{11}}$$

ways to choose all three link graphs such that no K_4 appears in the joint link.

Combining the above estimates and

$$n/4 \leq |Y| \leq n/2 + 2\sqrt{h(6\alpha)}n\,,$$

we obtain

$$\begin{aligned}
|\mathcal{F}''_n(\alpha,\beta) \setminus \mathcal{F}'''_n(\alpha,\beta)| &\leq \binom{n}{3} 2^{n-3} \cdot \binom{\binom{n}{2}}{\gamma n^2}^3 2^{3n^2/4} 2^{3\binom{|Y|}{2}-|Y|^2/2^{11}} |\mathcal{F}_{n-3}| \\
&\overset{(4.8)}{\leq} 2^{3\log n + n + 3h(2\gamma)n^2/2 + 9n^2/8 + \vartheta n^2/2 - n^2/2^{15}} |\mathcal{F}_{n-3}| \\
&\overset{(4.7)}{\leq} 2^{\delta(B_{n-2})+\delta(B_{n-1})+\delta(B_n)+\vartheta n^2 - n^2/2^{16}} |\mathcal{F}_{n-3}| \overset{(4.6)}{=} 2^{\delta(B_{n-2})+\delta(B_{n-1})+\delta(B_n)-\vartheta n^2} |\mathcal{F}_{n-3}| \\
&\overset{(4.9)}{\leq} 2^{-\vartheta n^2} \cdot 2^{n-3} \cdot |\mathcal{B}_n| \cdot (1 + 2^{n_0^2(n-3)-c(n-3)^2}) \leq |\mathcal{B}_n|(2^{-\vartheta n^2/2} + 2^{n_0^2 n - cn^2 - \vartheta n^2/2})\,,
\end{aligned}$$

where we used (4.3) for the penultimate inequality and $n_0 \geq 14/\vartheta$ and $c \leq 1$ for the last inequality. Finally, we derive the required upper bound on $|\mathcal{F}_n|$

$$\begin{aligned}
|\mathcal{F}_n| &\leq |\mathcal{F}_n \setminus \mathcal{F}'_n(\alpha)| + |\mathcal{F}'_n(\alpha) \setminus \mathcal{F}''_n(\alpha,\beta)| + |\mathcal{F}''_n(\alpha,\beta) \setminus \mathcal{F}'''_n(\alpha,\beta)| + |\mathcal{B}_n| \\
&\leq 2^{e(B_n)-c'n^3} + 2^{e(B_n)-c''n^3} + |\mathcal{B}_n|(2^{-\vartheta n^2/2} + 2^{n_0^2 n - cn^2 - \vartheta n^2/2}) + |\mathcal{B}_n| \\
&\leq |\mathcal{B}_n|(1 + 4 \cdot 2^{n_0^2 n - \vartheta n^2/2}) \leq |\mathcal{B}_n|(1 + 2^{n_0^2 n - cn^2}).
\end{aligned}$$

□

4.2.5 An example

While studying different classes of F-free hypergraphs, the reader might have noticed that $|\mathcal{F}'_n(\alpha)|/|\mathcal{B}_n|, |\mathcal{F}''_n(\alpha,\beta)|/|\mathcal{B}_n| \leq 2^{-\Omega(n^3)}$, while $|\mathcal{F}'''_n(\alpha,\beta)|/|\mathcal{B}_n| \leq 2^{-\Omega(n^2)}$, which in the summary led us to

$$\frac{|\mathrm{Forb}(n,F)| - |\mathcal{B}_n|}{|\mathcal{B}_n|} \leq 2^{-\Omega(n^2)}.$$

Below we construct $|\mathcal{B}_n|2^{-O(n^2)}$ non-bipartite Fano-free hypergraphs, thus leading to

$$(|\mathrm{Forb}(n,F)| - |\mathcal{B}_n|)/|\mathcal{B}_n| = 2^{-\Theta(n^2)}. \tag{4.10}$$

The rough idea is to take a random balanced bipartite hypergraph on $n-9$ vertices, and then add some additional hyperedges incident to some of the remaining 9 vertices, such that almost surely the resulting hypergraph is not bipartite. A routine application of Chernoff's inequality gives then (4.10).

Let H' be a random balanced bipartite 3-uniform hypergraph with classes A, B such that $|A| + |B| = n - 9$. Let $v_1, \ldots, v_9$ be new vertices, let $G_1, \ldots, G_6 \in \mathcal{G}(A, 1/2)$, i.e. G_i's are random graphs on the vertex set A. We define the hypergraph H to have

$$A \dot{\cup} B \dot{\cup} \{v_1, \ldots, v_9\}$$

as a vertex set, and we define the hyperedge set of H to be:

$$E(H) := E(H') \dot{\cup} \{e \cup \{v_i\} \colon e \in E(G_i), i \in [6]\} \dot{\cup} \\ \{\{v_1, v_2, v_7\}, \{v_3, v_4, v_8\}, \{v_5, v_6, v_9\}, \{v_7, v_8, v_9\}\}.$$

Note that $H - v_7v_8v_9$ is bipartite, and moreover, since the hyperedges incident with v_7, v_8 and v_9 are disjoint, H must be Fano-free. It is easy to show that with high probability, H is not bipartite. In fact, w.h.p. a proper coloring of $H - v_7v_8v_9$ is unique (up to permutation of the two colors). Since the probability space consists of

$$2^{e(B_{n-9}) + 6\binom{(n-9)/2}{2}}$$

elements, the inequality (4.10) follows from the bounds on $\mathcal{B}_n$ (4.3) and Theorem 1.7.

4.3 Coloring Fano-free (and bipartite) 3-uniform hypergraphs in polynomial expected time

4.3.1 Algorithm for coloring Fano-free hypergraphs

Below we first present the simple algorithm Color(H) which will be based on the subroutine Partition(H, α).

Obviously, Color(H) finds a proper coloring of H. We will show that Step 2 has a running time of $O(n^5 \log^2 n)$ for all H. Hence for proving Theorem 1.8 it suffices to show that there exists an $\alpha > 0$ such that Step 5 of the algorithm will be executed for at most $2^{-n \log_2 n}|\mathrm{Forb}(n,F)|$ 3-uniform hypergraphs from Forb(n, F), .

The subroutine Partition(H, α) finds a *locally minimal* partition $X_H \dot{\cup} Y_H = V(H)$, i.e., a partition for which $e(X_H) + e(Y_H)$ cannot be decreased by moving a single vertex from one class to another. Moreover, we will show later that for "most" 3-uniform hypergraphs H from Forb(n, F) the algorithm Partition(H, α) outputs a partition with the additional property $e(X_H) + e(Y_H) < \alpha n^3$.

Algorithm 1: Color (H)

Input: H from $\mathrm{Forb}(n, F)$
Output: Proper coloring of H

1 Choose $\alpha > 0$ appropriately (see Lemma 4.20);
2 $(X, Y) \leftarrow$ Partition (H, α);
3 **if** $e(X) + e(Y) = 0$ **then**
4   output 2-coloring corresponding to (X, Y);
5 **else**
6   try all $n^n = 2^{n \log_2 n}$ possible colorings and output the one that minimizes the number of colors used;

Algorithm 2: Partition (H, α)

Input: $H \in \mathrm{Forb}(n, F)$, $\alpha > 0$
Output: Locally minimal vertex partition of H: $V = X_H \dot{\cup} Y_H$

1 Choose $\varepsilon := \varepsilon(\alpha)$ and $\eta := \eta(\alpha)$ appropriately (see Lemma 4.16);
2 Apply Regularize$(H, \varepsilon, \lceil 1/\varepsilon \rceil)$ and obtain an ε-regular partition $V_1, \dots, V_t$;
3 Define cluster hypergraph $H(\eta)$ with densities at least η;
4 Find partition $A \dot{\cup} B$ of $V(H(\eta))$ which minimizes $e_{H(\eta)}(A) + e_{H(\eta)}(B)$;
5 Set $W_1 := \cup_{a \in A} V_a$ and $W_2 := \cup_{b \in B} V_b$;
6 **while** $\exists w \in W_i$ *such that* $\deg_{W_i}(w) > \deg_{W_{[2] \setminus \{i\}}}(w)$ **do**
7   move w to $W_{[2] \setminus \{i\}}$;
8 Output (W_1, W_2);

In Step 2 the algorithm Regularize(H, ε, t) will be used as a subroutine. This algorithm, due to Czygrinow and Rödl [CR00], finds an ε-regular partition of a 3-uniform hypergraph H on n vertices and at least t_0 many clusters in time $O(n^5 \log^2 n)$. Step 4 requires only constant time, that depends on ε only. The "while"-loop also requires at most $O(n^5)$ steps, as the update step requires at most $O(n^2)$ operations and the "while"-loop terminates after at most $\binom{n}{3}$ executions. Indeed, in every loop we increase the cut and there are at most $\binom{n}{3}$ hyperedges in H.

4.3.2 Overview of the analysis

So far we have presented our coloring algorithm Color(H) and it is left to show that there exist appropriate choices for α and for ε and η inside the subroutine Partition(H, α) which yield the claimed running time. More precisely, we will show that for sufficiently small α, ε and η the proportion of hypergraphs H in $\mathrm{Forb}(n, F)$ for which Step 6 in Color(H) is required, is "negligible".

In the main part of the proof we show that there are at most $2^{-\Omega(n^2)}|\mathrm{Forb}(n, F)|$ such hypergraphs in $\mathrm{Forb}(n, F)$. To prove this, we study structural properties of a typical H

from Forb(n, F). Our analysis builds on the ideas from previous section, Section 4.2. We will introduce a chain of subsets of Forb(n, F) such that all members of them possess certain "typical" properties. More precisely, we study the following chains of subsets of $\mathcal{F}_n$:

$$\mathcal{F}_n \supseteq \hat{\mathcal{F}}'_n(\alpha) \supseteq \hat{\mathcal{F}}''_n(\alpha, \beta) \supseteq \hat{\mathcal{F}}'''_n(\alpha, \beta),$$

and

$$\mathcal{F}_n \supseteq \mathcal{B}_n \supseteq \hat{\mathcal{F}}'''_n(\alpha, \beta).$$

The first subset $\hat{\mathcal{F}}'_n(\alpha)$ consists of those members, that admit at least one locally minimal partition (X_H, Y_H) with the properties $e(X_H) + e(Y_H) < \alpha n^3$ and $|X_H| \approx |Y_H|$. Using the algorithmic version of weak hypergraph regularity lemma, Theorem 4.2, and Theorem 4.1, we will show that for most of the members from $\hat{\mathcal{F}}'_n(\alpha)$ the algorithm Partition(H, α) finds a locally minimal partition for given α. Therefore, additionally, we obtain that most of the hypergraphs from Forb(n, F) lie in $\hat{\mathcal{F}}'_n(\alpha)$.

The further analysis proceeds as follows. We introduce two more proper subsets of Forb(n, F): $\hat{\mathcal{F}}''_n(\alpha, \beta)$ and $\hat{\mathcal{F}}'''_n(\alpha, \beta)$ which describe two further "useful" properties of almost all Fano-free hypergraphs on n vertices. The family $\hat{\mathcal{F}}''_n(\alpha, \beta)$ contains those members from $\hat{\mathcal{F}}'_n(\alpha)$ which are "dense everywhere" in the sense that whenever we take three disjoint subsets of vertices, say W_1, W_2, W_3, not all of them contained in X_H or Y_H (for *any* locally minimal partition satisfying properties from $\hat{\mathcal{F}}'_n(\alpha)$), the number of hyperedges that run between them will be at least $d|W_1||W_2||W_3|$ for some positive constant $d > 0$. Moreover, every vertex will have small degree in its own partition class (i.e. X_H or Y_H). Thus, essentially, there exists "only one" locally minimal partition. For members of the last class $\hat{\mathcal{F}}'''_n(\alpha, \beta)$ we demand that the joint link of every set of 3 vertices of any of the two partition classes X_H and Y_H must contain a K_4. We then deduce that the last property implies in fact bipartiteness. As a seemingly surprising fact, we obtain, that for almost all members from Forb(n, F) *any* locally minimal partition for some appropriate α already satisfies $e(X_H) + e(Y_H) = 0$.

4.3.3 Proof of Theorem 1.8

Below we give proper definitions of the classes described above and we state the lemmas that relate the sizes of these hypergraph classes. The proofs of the corresponding statements are given in the next section, Section 4.3.4. First we recall the definition of a *locally minimal* partition. A vertex partition $X_H \dot\cup Y_H$ of $V(H)$ is locally minimal if $e(X', Y') \geq e(X_H, Y_H)$ for all partitions $X' \dot\cup Y'$ of $V(H)$ with $|X' \Delta X_H| \leq 1$. Furthermore, we say a partition is α-good, if $e(X_H) + e(Y_H) < \alpha n^3$ and $|X_H|, |Y_H| < n/2 + 2\sqrt{h(6\alpha)}n$. The first class $\hat{\mathcal{F}}'_n(\alpha)$ of Fano-free hypergraphs is defined as follows.

Definition 4.15. *Let $\alpha > 0$ and $n \in \mathbb{N}$. We set*

$$\hat{\mathcal{F}}'_n(\alpha) = \{H \in \mathcal{F}_n \colon \exists \text{ a locally minimal } \alpha\text{-good partition } V = X_H \dot\cup Y_H\}\,.$$

Lemma 4.16. *For every $\alpha \in (0, \frac{1}{12})$ there exist (computable) constants $c', \varepsilon, \eta > 0$ and an integer n'_0 such that for all $n \geq n'_0$ the algorithm Partition(H, α) finds for all but at* *most* $2^{e(B_n)-c'n^3}$ *hypergraphs* $H \in \mathcal{F}_n$ *a locally minimal partition* $X_H \dot\cup Y_H$ *of its vertex set with the following two properties:*

- $e(X_H) + e(Y_H) < \alpha n^3$,
- $|X_H|, |Y_H| < n/2 + 2\sqrt{h(6\alpha)}n$.

In particular, we have:

$$|\mathcal{F}_n \setminus \hat{\mathcal{F}}'_n(\alpha)| < 2^{e(B_n)-c'n^3}.$$

Next we define the subfamily of "everywhere dense" hypergraphs from $\hat{\mathcal{F}}'_n(\alpha)$.

Definition 4.17. *For* $\alpha, \beta > 0$ *and* $n \in \mathbb{N}$ *let* $\hat{\mathcal{F}}''_n(\alpha, \beta)$ *denote the family of those hypergraphs* $H \in \hat{\mathcal{F}}'_n(\alpha)$*, such that for* any *locally minimal* α*-good partition* $X_H \dot\cup Y_H$ *of* $V(H)$ *the following condition holds.*

For any pairwise disjoint sets $W_1 \subset X_H$*,* $W_2 \subset Y_H$ *and* $W_3 \subset Z_H$*, where* $Z_H \in \{X_H, Y_H\}$*, with* $|W_i| \geq \beta n$ *for* $i = 1, 2, 3$ *we have*

$$e_H(W_1, W_2, W_3) \geq \frac{1}{4}|W_1||W_2||W_3|.$$

The proof of the following lemma follows the lines of the proofs of Lemmas 4.10 and 4.11. and we only give a sketch of this proof below.

Lemma 4.18. *For every* $\gamma > 0$ *there exist (computable) constants* $\alpha, \beta, c'' > 0$ *and an integer* n_0 *such that for every* $n \geq n_0$ *and* $H \in \hat{\mathcal{F}}''_n(\alpha, \beta)$*, we have:*

- $|\hat{\mathcal{F}}'_n(\alpha) \setminus \hat{\mathcal{F}}''_n(\alpha, \beta)| < 2^{e(B_n)-c''n^3}$.
- $\Delta(H[X_H]), \Delta(H[Y_H]) < \gamma n^2$ *for any locally minimal* α*-good partition* $X_H \dot\cup Y_H$*,*

Proof (sketch). The first part follows from a simple counting argument or, alternatively, from Chernoff's estimate (see, e.g. 4.10).

To show the second property one simply regularizes the link of a vertex of high degree with the Szemerédi regularity lemma for graphs [Sze78]. Inside the classes X_H and Y_H one then identifies (ε', d')-regular pairs for appropriate $\varepsilon' = \varepsilon'(\gamma)$ and $d' = d'(\gamma)$, and together with the fact that (X_H, Y_H) is dense everywhere one can find a copy of the Fano plane by appealing to the Key-Lemma, Theorem 4.4.

□

We finally define the last subclass of Fano-free hypergraphs.

Definition 4.19. *For* $\alpha, \beta > 0$ *and* $n \in \mathbb{N}$*, let* $\hat{\mathcal{F}}'''_n(\alpha, \beta)$ *denote the family of those hypergraphs* $H \in \hat{\mathcal{F}}''_n(\alpha, \beta)$*, such that for* any *locally minimal* α*-good partition* $X_H \dot\cup Y_H$ *of* $V(H)$ *the following holds.*

For all triples $z_1, z_2, z_3 \in Z$ *of vertices with* $Z \in \{X_H, Y_H\}$ *we have* $L_Q(z_1) \cap L_Q(z_2) \cap L_Q(z_3) \supseteq K_4$*, where* $\{Q, Z\} = \{X_H, Y_H\}$*. In other words, we require that the common link of any triple from* X_H *or* Y_H *contains a copy of* K_4 *in the other vertex class.*

It follows directly from the definition, that every $H \in \hat{\mathcal{F}}'''_n(\alpha, \beta)$ is bipartite, i.e., $\hat{\mathcal{F}}'''_n(\alpha, \beta) \subseteq \mathcal{B}_n$. Otherwise, any hyperedge e, say in X_H, together with the K_4 in Y_H, which lies in the common link of the vertices of e would span a copy of the hypergraph of the Fano plane. We also note that we could replace K_4 in the definition of $\hat{\mathcal{F}}'''_n(\alpha, \beta)$ by a 1-factor of K_4 that is created by the union of the links of any three vertices.

To obtain a bound on $|\hat{\mathcal{F}}''_n(\alpha, \beta) \setminus \hat{\mathcal{F}}'''_n(\alpha, \beta)|$ we estimate in at most how many ways one can construct a Fano-free hypergraph from $\hat{\mathcal{F}}''_n(\alpha, \beta) \setminus \hat{\mathcal{F}}'''_n(\alpha, \beta)$, i.e. a Fano-free hypergraph with a locally minimal partition and a hyperedge inside it.

Lemma 4.20. *There exist (computable) constants $\alpha, \beta, c > 0$ and an integer n_0, such that for every $n \geq n_0$ we have*

$$|\hat{\mathcal{F}}'_n(\alpha) \setminus \hat{\mathcal{F}}'''_n(\alpha, \beta)| \leq 2^{e(B_n) - cn^2}.$$

We remark that the bound in Lemma 4.20 is considerably weaker than those in Lemmas 4.18 and 4.16 having only $-cn^2$ in the exponent instead of $-cn^3$. This is however necessary, as shown in Section 4.2.5.

Proof of Theorem 1.8. We first apply Lemma 4.20 and obtain constants α, β and c. Then Lemma 4.16 applied with α returns ε, η and c'. Below we show that these constants α, ε and η are suitable choices in the algorithms Color and Partition.

Indeed for these choices of α, ε and η Lemma 4.16 asserts that the partition $X_H \dot\cup Y_H$ provided by Partition(H, α) is locally minimal and α-good for all but $2^{e(B_n) - c'n^3}$ hypergraphs $H \in \mathcal{F}_n$. Moreover, due to Lemma 4.20 this partition satisfies $e(X_H) + e(Y_H) = 0$ for all $H \in \hat{\mathcal{F}}'''_n(\alpha, \beta)$, i.e. this partition is a correct 2-coloring of H. Finally, it follows from Lemma 4.16 and Lemma 4.20 that Step 6 is only considered for at most

$$2^{e(B_n) - c'n^3} + 2^{e(B_n) - cn^2}$$

Fano-free hypergraphs. □

4.3.4 Proofs of Lemmas 4.16 and 4.20

Proof of Lemma 4.16. The proof of Lemma 4.16 combines the weak hypergraph regularity lemma with the stability theorem for Fano-free hypergraphs applied to the cluster hypergraph.

Let $\lambda = \lambda(\alpha/2)$ and $n_0(\alpha/2)$ be given by Theorem 4.1. We may assume $\lambda < 16h(6\alpha)$. We set

$$c' = \frac{\lambda}{17},$$

and we choose η such that $\lambda > (16/3)h(6\eta)$ and $\eta \leq \alpha/2$. Finally let $\varepsilon = \varepsilon(\eta/2) \leq \eta/2$ be given by a version of counting lemma, Lemma 2.10. Set $t_0 = \max\{1/\varepsilon, n_0(\alpha/2)\}$ and let n be sufficiently large, in particular, set $n'_0 \gg \max\{T_0, n_0\}$, where T_0 and n_0 are given by the weak regularity lemma, Theorem 4.2. For the main steps of the proof it is sufficient to keep in mind that

$$0 < 1/t_0 \leq \varepsilon \leq \eta \ll \lambda \ll \alpha.$$

We may assume in the following that t divides n, and thus $|V_i| = n/t$ for all $i = 1, \dots, t$.

We will upper bound the number of hypergraphs for which Partition(H, α) fails to produce a locally minimal α-good partition in two steps. More formally, we consider the subset $\widetilde{\mathcal{F}'_n}(\alpha)$ which consists of those hypergraphs H from $\mathcal{F}_n$, for which Partition(H, α) returns a locally minimal α-good partition. Thus, our aim is to show that $|\mathcal{F}_n \setminus \widetilde{\mathcal{F}'_n}(\alpha)|$ is at most $2^{e(B_n)-c'n^3}$. Our proof has two steps.

In the first step we bound the number of hypergraphs H that have $e(X_H) + e(Y_H) \geq \alpha n^3$ for *every* locally minimal partition (X_H, Y_H). In the second step we show that for most of the hypergraphs H every locally minimal partition $X_H \dot{\cup} Y_H$ with $e(X_H) + e(Y_H) < \alpha n^3$ will also satisfy:

$$\max\{|X_H|, |Y_H|\} < \frac{n}{2} + 2\sqrt{h(6\alpha)}n\,.$$

Here and in the following (X_H, Y_H) will stand for a locally minimal partition, and unless it is specified otherwise, it will stand for an arbitrary locally minimal partition.

Step 1.

Consider a hypergraph $H \in \mathcal{F}_n$ satisfying

$$e(X_H) + e(Y_H) \geq \alpha n^3 \tag{4.11}$$

for every locally minimal partition. We apply the weak regularity lemma, Theorem 4.2, with parameters ε and t_0. Firstly, we estimate the number of hyperedges, which are contained in the "uncontrolled" part of the regular partition:

- the number of hyperedges intersecting at most two of the clusters is at most

$$t\binom{n/t}{2}n < \frac{1}{2t}n^3,$$

- the number of hyperedges contained in irregular triples is at most

$$\varepsilon\binom{t}{3}\left(\frac{n}{t}\right)^3 < \frac{\varepsilon}{6}n^3,$$

- the number of hyperedges that are contained in ε-regular triples of density less than η is at most

$$\eta\left(\frac{n}{t}\right)^3\binom{t}{3} < \frac{\eta}{6}n^3.$$

Thus, the number of discarded edges is less than ηn^3.

Secondly, consider the resulting cluster-hypergraph $H(\eta)$. It must be Fano-free as otherwise Lemma 2.10 would imply that H also contains a copy of the hypergraph of the Fano plane. We assumed that $e(X_H) + e(Y_H) \geq \alpha n^3$, so we can bound the number of hyperedges in $H(\eta)$ from above by $(1-\lambda)t^3/8$. Otherwise, Theorem 4.1 would yield the existence of a partition of $V_1, \dots, V_t$ into disjoint sets X and Y with $e_{H(\eta)}(X) + e_{H(\eta)}(Y) < \alpha t^3/2$. Defining a partition of $V(H)$ into the following two sets

$$A = \bigcup_{U \in X} U \quad \text{and} \quad B = \bigcup_{W \in Y} W,$$

with

$$e_H(A) + e_H(B) < \eta n^3 + \frac{\alpha}{2} t^3 \binom{n}{t}^3 \leq \alpha n^3\,,$$

which yields a contradiction to $e(X_H) + e(Y_H) \geq \alpha n^3$. Note that shifting vertices until a *locally minimal* partition is found only decreases $e(X_H) + e(Y_H)$.

Now we are able to bound the number of hypergraphs $H \in \mathcal{F}_n$ with $e(X_H) + e(Y_H) \geq \alpha n^3$ for every locally minimal partition $X_H \dot\cup Y_H$ from above by calculating the total possible number of ε-regular partitions together with all possible cluster-hypergraphs associated with them and all possible hypergraphs that could give rise to such a particular cluster-hypergraph. This way we get at most

$$\begin{aligned}\sum_{t=t_0}^{T_0} t^n \cdot 2^{\binom{t}{3}} \cdot 2^{(1-\lambda)\frac{t^3}{8}(\frac{n}{t})^3} \cdot \left(\sum_{j=0}^{\eta n^3 - 1} \binom{\binom{n}{3}}{j} \right) &\leq T_0^{n+1} \cdot 2^{\binom{T_0}{3}} \cdot 2^{(1-\lambda)n^3/8} \cdot \binom{\binom{n}{3}}{\eta n^3} \\ &\leq 2^{(n+1)\log T_0 + \binom{T_0}{3} + n^3/8 - \lambda n^3/8 + h(6\eta)n^3/6} < 2^{n^3/8 - \lambda n^3/16}\,,\end{aligned}$$

hypergraphs with property (4.11) for sufficiently large n, due to the choice of η.

Step 2.

We now estimate the number of those hypergraphs H which have a locally minimal partition (X_H, Y_H) with $e(X_H) + e(Y_H) < \alpha n^3$, but $\max\{|X_H|, |Y_H|\} \geq n/2 + 2\sqrt{h(6\alpha)}n$. First we upper bound $e(X_H, Y_H)$ for such a hypergraph H by

$$e(X_H, Y_H) \leq |X_H| \binom{|Y_H|}{2} + |Y_H| \binom{|X_H|}{2} < \frac{n}{2} |X_H||Y_H| < \frac{n^3}{8} - 2h(6\alpha)n^3.$$

Note that there are at most 2^n possible partitions of $V(H)$, and since less than αn^3 hyperedges are completely contained in X_H and Y_H, those hyperedges can be chosen in at most

$$\sum_{i=0}^{\alpha n^3 - 1} \binom{\binom{n}{3}}{i} \leq \binom{\binom{n}{3}}{\alpha n^3}$$

ways. Finally, as we assumed that our partitions are "unbalanced" we estimate the number of possible choices of hyperedges between X_H and Y_H by $2^{n^3/8-2h(6\alpha)n^3}$. Altogether we get, that there are at most

$$2^n \cdot \binom{\binom{n}{3}}{\alpha n^3} \cdot 2^{n^3/8-2h(6\alpha)n^3} \leq 2^{n+h(6\alpha)n^3/6+n^3/8-2h(6\alpha)n^3} \leq 2^{n^3/8-h(6\alpha)n^3}$$

hypergraphs H with $e(X_H)+e(Y_H) < \alpha n^3$ and

$$\max\{|X_H|, |Y_H|\} \geq \frac{n}{2} + 2\sqrt{h(6\alpha)}n.$$

Combining Step 1 and 2 we obtain:

$$|\mathcal{F}_n \setminus \hat{\mathcal{F}}'_n(\alpha)| \leq |\mathcal{F}_n \setminus \widetilde{\mathcal{F}}'_n(\alpha)| \leq 2^{n^3/8-\lambda n^3/16} + 2^{n^3/8-h(6\alpha)n^3} < 2^{n^3/8-\lambda n^3/16+1},$$

since $h(6\alpha) > \lambda/16$. Due to $n^3/8 - e(B_n) \leq n^2/4 + O(n)$ and the choice of $c' = \lambda/17$, the lemma follows for sufficiently large n. □

Proof of Lemma 4.20. We set

$$\vartheta = 2^{-17} \quad \text{and} \quad c = \frac{\vartheta}{4} \tag{4.12}$$

and choose $\gamma > 0$ such that

$$3h(2\gamma) < \vartheta\,. \tag{4.13}$$

Let α and $\beta > 0$ be given by Lemma 4.18. We may also assume that

$$3\sqrt{h(6\alpha)} + 6h(6\alpha) < \vartheta/2\,, \tag{4.14}$$

as choosing α smaller we will only have to eventually increase n_0. Again, it is sufficient to keep in mind that

$$0 < \alpha, \beta \ll \gamma \ll \vartheta = 2^{-17}\,.$$

Due to Lemma 4.18 we have

$$|\hat{\mathcal{F}}'_n(\alpha) \setminus \hat{\mathcal{F}}''_n(\alpha,\beta)| \leq 2^{e(B_n)-c''n^3}, \tag{4.15}$$

and we estimate $|\hat{\mathcal{F}}''_n(\alpha,\beta) \setminus \hat{\mathcal{F}}'''_n(\alpha,\beta)|$ now.

Let $H \in \hat{\mathcal{F}}''_n(\alpha,\beta) \setminus \hat{\mathcal{F}}'''_n(\alpha,\beta)$ and $X_H \dot\cup Y_H$ be an arbitrary locally minimal α-good partition. Consider a subset $S \in \binom{X_H}{3} \dot\cup \binom{Y_H}{3}$. Deleting S from $V(H)$, we obtain a Fano-free hypergraph H' on $n-3$ vertices, where $V(H') = [n] \setminus S$. Note that for every $H \in \hat{\mathcal{F}}''_n(\alpha,\beta) \setminus \hat{\mathcal{F}}'''_n(\alpha,\beta)$ there exists a hypergraph $H' \in \mathcal{F}_{n-3}$ such that H can be reconstructed from H' in the following way. For $H' \in \mathcal{F}_{n-3}$ we choose a set S of 3 vertices, which we "connect" in an appropriate manner, so that the resulting hypergraph is in $\hat{\mathcal{F}}''_n(\alpha,\beta) \setminus \hat{\mathcal{F}}'''_n(\alpha,\beta)$.

We can choose the set S, the partition of H' and the set which contains S in at most

$$\binom{n}{3}2^{n-3}$$

ways. Since $H \in \hat{\mathcal{F}}''_n(\alpha,\beta)$ we infer from Lemma 4.18 that every vertex in S has at most γn^2 neighbors in its own partition class. This again bounds the number of ways for choosing these hyperedges by

$$\left(\sum_{j=0}^{\gamma n^2-1}\binom{\binom{n}{2}}{j}\right)^3 \le \binom{\binom{n}{2}}{\gamma n^2}^3 .$$

For every vertex in S we have at most $2^{n^2/4}$ possibilities for choosing hyperedges with one more vertex in the same partition as S and the other vertex in the other partition class, this gives us at most

$$2^{3n^2/4}$$

ways to choose that type of hyperedges. The last estimate concerns the number of ways we can connect our triple S to the other partition class, say Y, without creating any single copy of K_4, which is contained in the joint link of the vertices from S. Here we use Lemma 4.13. For every vertex v in S we can choose its link graph $L_Y(v)$ in at most $2^{\binom{|Y|}{2}}$ ways. However, since the joint link of three vertices in S contains no K_4, we infer from Lemma 4.13, that there are at most

$$2^{3\binom{|Y|}{2}}\exp(-2^{-11}|Y|^2) < 2^{3\binom{|Y|}{2}-|Y|^2/2^{11}}$$

ways.

Combining the above estimates and

$$n/4 \le |Y| \le n/2 + 2\sqrt{h(6\alpha)}n\,,$$

we obtain

$$\begin{aligned}
|\hat{\mathcal{F}}''_n(\alpha,\beta)\setminus\hat{\mathcal{F}}'''_n(\alpha,\beta)| &\le \binom{n}{3}2^{n-3}\cdot\binom{\binom{n}{2}}{\gamma n^2}^3 2^{3n^2/4}2^{3\binom{|Y|}{2}-|Y|^2/2^{11}}|\mathcal{F}_{n-3}| \\
&\overset{(4.14)}{\le} 2^{3\log n+n+3h(2\gamma)n^2/2+9n^2/8+\vartheta n^2/2-n^2/2^{15}}|\mathcal{F}_{n-3}| \\
&\overset{(4.13)}{\le} 2^{\delta(B_{n-2})+\delta(B_{n-1})+\delta(B_n)+\vartheta n^2-n^2/2^{16}}|\mathcal{F}_{n-3}| \\
&\overset{(4.12)}{=} 2^{\delta(B_{n-2})+\delta(B_{n-1})+\delta(B_n)-\vartheta n^2}|\mathcal{F}_{n-3}| \overset{(1.5)}{\le} 2^{-\vartheta n^2}\cdot 2^{n-2}\cdot|\mathcal{B}_n|.
\end{aligned}$$

Since

$$|\mathcal{F}_n| \overset{(1.5)}{\le} 2|\mathcal{B}_n| \le 2^{e(B_n)+n+1},$$

it follows from (4.15), that

$$|\hat{\mathcal{F}}'_n(\alpha) \setminus \hat{\mathcal{F}}'''_n(\alpha, \beta)| \leq 2^{e(B_n)-cn^2}$$

due to the choice of c. □

4.4 Concluding remarks

4.4.1 Induced case

In this chapter we discussed the family $\mathrm{Forb}(n, F)$, containing all labeled hypergraphs without a copy of the Fano plane and obtained a result on the structure of almost all members of this set. However, we did not consider the induced case, i.e. $\mathrm{Forb}^{\mathrm{ind}}(n, F)$, the family of labeled hypergraphs on n vertices not containing an induced copy of a hypergraph F. In the graph case much of the work was done by Prömel and Steger [PS91, PS92a, PS92d], who studied $|\mathrm{Forb}^{\mathrm{ind}}(n, F)|$ for a single copy of any graph F, and consequently by Alekseev [Ale92] and by Bollobás and Thomason [BT97], who investigated $|\mathrm{Forb}^{\mathrm{ind}}(n, \mathcal{F})|$ for arbitrary families of graphs. The asymptotic bounds obtained on those families are of the following form:

$$|\mathrm{Forb}^{\mathrm{ind}}(n, \mathcal{F})| \leq 2^{(1-1/(\chi_{\mathrm{col}}(\mathcal{F})-1))\binom{n}{2}+o(n^2)},$$

where $\chi_{\mathrm{col}}(\mathcal{F})$ is the so-called coloring number introduced by Prömel and Steger [PS93], which plays a similar role that the chromatic number does for the non-induced case, cf. (1.3). These results were extended to 3-uniform hypergraphs by Kohayakawa, Nagle and Rödl [KNR03] and to general k-uniform hypergraphs by Dotson and Nagle [DN09]. Quite recently, Alon, Balogh, Bollobás and Morris [ABBM] studied the structure of almost all graphs in a hereditary property and generalized the results of Alekseev and of Bollobás and Thomason. Balogh and Butterfield [BB] proved some further, more finer structural results on some hereditary properties of graphs. It would be interesting to prove some results for hypergraphs of the "almost all"-type, similar to [PS91, PS92a, BB].

4.4.2 Refining the structure of members from $\mathrm{Forb}(n, F)$

We believe that with essentially the same methods one could prove results similar to Theorem 1.7 for other linear k-uniform hypergraphs (instead of the hypergraph of the Fano plane), which contain at least one color-critical hyperedge and which admit a stability result similar to Theorem 4.1. Here by a color-critical hyperedge we mean a hyperedge e of H such that $\chi(H-e) < \chi(H)$. Natural candidates of such hypergraphs are $F^k_{\ell+1}$ and $H^k_{\ell+1}$, studied in Chapter 5 in connection with a related problem on counting restricted edge colorings.

Another possible direction is to improve for various (not necessarily linear) hypergraphs F an upper bound given by

$$|\mathrm{Forb}(n, F)| \leq 2^{\mathrm{ex}(n,F)+o(n^k)}.$$

This was made recently by Balogh and Mubayi [BMa, BMb], who used the hypergraph regularity lemma of Frankl and Rödl [FR02] to prove structural results of type "almost all" for T_3, the 3-uniform generalized triangle and for triple systems with independent neighborhoods.

Most of the recent proofs that study $\mathrm{Forb}(n, F)$ employ at the first step some version of (hyper-)graph regularity lemma, then subsequent study of subclasses that possess some nice properties that are shared by almost all members from $\mathrm{Forb}(n, F)$ often leads to precise characterizations [EKR76, KPR87, PS92b, PS09a, BMa, BMb]. However, as the extremal theory for hypergraphs is richer and less understood than for graphs, only for few hypergraphs extremal results are known.

5 Restricted edge colorings of hypergraphs

Suppose one is given a fixed graph F and r colors. Then in at most how many ways can one color the edges of a graph on n vertices such that no monochromatic copy of F is created? And how do graphs attaining the maximum look like?

This was resolved by Yuster for 2 colors for F being a graph triangle K_3 [Yus96], and by Alon, Balogh, Keevash and Sudakov [ABKS04] for general graphs as discussed in Section 1.2.3 of the Introduction.

Here we study the "same" function for hypergraphs, applying both versions (the weak and the strong one) of hypergraph regularity lemmas. First we show an approximate structural result when the number of colors is 2 or 3, this is Theorem 1.11 obtained together with Hanno Lefmann and Mathias Schacht [LPS], a special case of it was also proved together with the fourth author, Vojtěch Rödl, in [LPRS09]. Both theorems are presented in the section below.

Then we improve for the Fano plane, for the generalized 3- and 4-uniform triangles, for the expanded complete graphs and for the Fan(k)-hypergraphs these approximate results to exact ones. For these hypergraphs, it always turns out that for 2 and 3 colors the maximizers are the extremal hypergraphs while for $r \geq 4$ they are no longer optimal. The mentioned results are Theorem 1.12, Theorem 1.13, Theorem 1.14 and Theorem 1.15. The "exact" results are presented in Section 5.2, while the hypergraphs with more edge colorings for $r \geq 4$ are constructed in Section 5.3. Finally, we present in Section 5.4 a general upper bound on $c_{r,F}(n)$.

5.1 Structure of hypergraphs with many restricted edge colorings

Given a k-uniform hypergraph F, which is s-stable. We study the structure of those (k-uniform) hypergraphs H on n vertices that admit at least $r^{\mathrm{ex}(n,F)-o(n^k)}$ r-edge colorings with no monochromatic F and $r = 2, 3$. It will turn out that, in the simplest case of 1-stability, H must be $o(1)$-close to the extremal hypergraph for F. Here we first prove a special case of Theorem 1.11, Theorem 5.1, for F being the hypergraph of the Fano plane. In some sense it is a "toy" example: it is a generalization of a corresponding theorem from [ABKS04] and in turn it easily, almost verbatim, generalizes to the special case of Theorem 1.11 for general stable linear hypergraphs F. On the other side we modify the original proof of Theorem 5.1 and this idea will lead us to the proof of Theorem 1.11, which is quite technical due to its use of the Theorems 2.16 and 2.17. Thus, Theorem 5.1 only serves as a case to make the key ideas clear.

5.1.1 Overview of the proof of Theorem 1.11

We first discuss the proof strategy from [ABKS04]. One splits the proof into two parts. In the first part the following structural result is obtained. If, say, a graph G on n vertices admits at least $r^{\mathrm{ex}(n,K_\ell)}$ many edge colorings without a monochromatic complete graph K_ℓ, then G looks almost like the Turán graph $T_{\ell-1}(n)$. To prove such a structural result one fixes some edge coloring of G without a monochromatic copy of F, and one applies the colored version of the regularity lemma to G, Theorem 2.3. Then, almost all edges are contained in ε-regular pairs of sufficiently large density. This allows to concentrate on the cluster graphs defined for every color. The idea now is to consider the new cluster graph R that consists of the ε-regular pairs in all colors. Furthermore, if one could apply to that cluster graph the stability result of Erdős and Simonovits [Sim68], Theorem 2.1, then this would give also a partition of the underlying graph, as then the number of the other ε-regular pairs that are contained inside the partition classes would be small. But the stability result need not be applicable at least for one particular coloring. However, one then derives a contradiction by bounding $c_{r,K_\ell}(G)$ from above by $r^{\mathrm{ex}(|V(G)|,K_\ell)-1}$.

In the second part of the argument one uses backward induction. Assuming that G is not the Turán graph, one consecutively removes vertices to derive an impossible fact about some subgraph $G' \subseteq G$. Namely, that $c_{r,K_\ell}(G') > r^{\binom{|V(G')|}{2}}$, which is clearly a contradiction, as any graph on $|V(G')|$ vertices may have at most $\binom{|V(G')|}{2}$ edges.

We generalize the first part of the argument for stable hypergraphs F. To get an approximate result for 3-uniform hypergraphs with F being the Fano plane, the weak hypergraph regularity lemma in conjunction with the counting lemma for linear hypergraphs, Lemma 2.6(also Lemma 2.10) is enough because F is linear. In this setting the notion of the cluster hypergraph is the same as for graphs.

For the general hypergraph case, we use the regularity lemma of Rödl and Schacht, Theorem 2.16 together with some form of the corresponding counting lemma, Theorem 2.17 proved also by these authors, see [RS07b, RS07a]. However, applications of this lemma yield partitions with a more complicated structure. In particular, there is no such simple notion of the cluster hypergraph as in the graph case. Nevertheless, a way to obtain a similar structural result is still to apply some appropriate stability result, but this time not to the cluster hypergraph, but to an underlying hypergraph, which arises if one ignores the colors in the cluster hypergraph. For graphs, this can be explained as follows. Consider the graph that arises from the cluster graph R if one ignores the colors. Then we can apply the stability result, unless some copy of K_ℓ suddenly appears. This is impossible due to the fact that then the edges of this copy must correspond to ε-regular pairs, which are ε-regular in *every* color. Thus, these pairs form a copy of K_ℓ in the cluster graph which is a contradiction.

5.1.2 Fano plane

Theorem 5.1 (Structural result for Fano plane)**.** *Let $r = 2$ or $r = 3$. Then for every $\alpha > 0$ there exists $n_0 = n_0(r, \alpha)$ such that every hypergraph $H = (V, E)$ on $n > n_0$ vertices with $c_{r,F}(H) \geq r^{e(B_n)}$ admits a partition of its vertex-set into $V(H) = X \dot\cup Y$*

such that $e(X) + e(Y) < \alpha n^3$.

Proof. We prove the theorem only for $r = 3$, as the proof for $r = 2$ is very similar. Let $\alpha > 0$ be given. Fix γ sufficiently small with $0 < \gamma < 1$ such that

$$133\gamma + 66h(6\gamma) < \frac{\alpha}{2} \quad \text{and} \quad 44\gamma + 22h(6\gamma) < \lambda(\alpha/2)\,, \tag{5.1}$$

where $\lambda(\alpha/2)$ is given by Theorem 4.1 and h is the entropy function (2.1). Note that such a γ exists, since $h(6\gamma) \to 0$ as $\gamma \to 0$. Let $\varepsilon = \varepsilon(\gamma) > 0$ with $\varepsilon < \gamma/2$ be such, that Lemma 2.10 is satisfied. Moreover, let $t_0 = \max\{1/\varepsilon, t'\}$, where t' is sufficiently large, so that (4.1) holds, i.e., $\mathrm{ex}(t, F) = e(B_t)$ for every $t \geq t'$, and so that Theorem 4.1 holds for $\alpha/2$ for all hypergraphs on at least t' vertices.

Let $T_0 = T_0(3, t_0, \varepsilon)$ and $N_0 = N_0(3, t_0, \varepsilon)$ be according to Theorem 2.8 and let $m_0 = m_0(\gamma)$ be according to Lemma 2.10. Finally, set $n_0 := \max\{N_0, T_0 \cdot m_0\}$.

Let $H = (V, E)$ be a hypergraph on $n \geq n_0$ vertices, which admits at least $3^{e(B_n)}$ Fano plane-free 3-colorings of the set of hyperedges. Let us denote the colors by red, blue and green.

Consider any fixed Fano plane-free 3-coloring of the set of hyperedges of H. By Theorem 2.8 for $r = 3$ there exists a positive integer $T_0 = T_0(3, t_0, \varepsilon)$ and there exists a partition $V(H) = V_1 \dot\cup \ldots \dot\cup V_t$ of the vertex set $V(H)$, $t_0 \leq t \leq T_0$, which is ε-regular with respect to each color class, where $|V_i| \leq \lceil n/t \rceil$, $1 \leq i \leq t$. To simplify the calculations, we assume in the following that $|V_i| = n/t \in \mathbb{N}$, $1 \leq i \leq t$. This does not change our asymptotic analysis.

Let $H_{\mathrm{red}}(\gamma), H_{\mathrm{blue}}(\gamma)$ and $H_{\mathrm{green}}(\gamma)$ be the corresponding cluster hypergraphs on the vertex set $[t] = \{1, \ldots, t\}$, i.e., $H_{\mathrm{col}}(\gamma)$ corresponds to all those hyperedges with color $\mathrm{col} \in \{\mathrm{red}, \mathrm{blue}, \mathrm{green}\}$, which are contained in ε-regular triples of density at least γ. By our assumption and by Lemma 2.10 each hypergraph $H_{\mathrm{col}}(\gamma)$ is Fano plane-free, hence each contains at most $\mathrm{ex}(t, F) = e(B_t)$ hyperedges.

We count the number of 3-colorings of the set of hyperedges, which yield the partition $V(H) = V_1 \dot\cup \cdots \dot\cup V_t$ of the vertex set and the cluster hypergraphs $H_{\mathrm{red}}(\gamma)$, $H_{\mathrm{blue}}(\gamma)$, and $H_{\mathrm{green}}(\gamma)$. To do so, first we bound from above the number of hyperedges $e \in E(H)$, which intersect some set V_i, $1 \leq i \leq t$, in at least two vertices, or are contained in a triple (V_i, V_j, V_k) which is not ε-regular, or for one color class are contained in a triple (V_i, V_j, V_k) of edge-density less than γ, $1 \leq i < j < k \leq t$.

The number of hyperedges $e \in E(H)$, which intersect one of the sets $V_1, \ldots, V_t$ in at least two vertices, is at most

$$t\binom{n/t}{2} n < \frac{1}{2t} n^3. \tag{5.2}$$

The number of hyperedges $e \in E(H)$, which are contained in one of the at most $3\varepsilon\binom{t}{3}$ ε-irregular triples (V_i, V_j, V_k), $1 \leq i < j < k \leq t$, is at most

$$3\varepsilon\binom{t}{3}\left(\frac{n}{t}\right)^3 < \frac{\varepsilon}{2} n^3. \tag{5.3}$$

The number of hyperedges $e \in E(H)$, which for one of the three color classes are contained in triples (V_i, V_j, V_k) of density less than γ, $1 \le i < j < k \le t$, is at most

$$3\binom{t}{3}\gamma\left(\frac{n}{t}\right)^3 < \frac{\gamma}{2}n^3. \tag{5.4}$$

With $t \ge t_0 \ge 1/\varepsilon$ and $\varepsilon < \gamma/2$, the total number of all these hyperedges is by (5.2)–(5.4) less than

$$\gamma n^3. \tag{5.5}$$

These hyperedges can be chosen in at most

$$\binom{\binom{n}{3}}{\gamma n^3} < \binom{n^3/6}{\gamma n^3} \le 2^{h(6\gamma)n^3/6} \tag{5.6}$$

ways – here we used (2.1)– and can be colored by red, blue or green in at most

$$3^{\gamma n^3} \tag{5.7}$$

ways.

Next we consider the set of all remaining hyperedges in H, i.e., those, which are contained in ε-regular triples (V_i, V_j, V_k) of density at least γ for every color class, $1 \le i < j < k$. If $\{i, j, k\}$ is a hyperedge in exactly s, $1 \le s \le 3$, of the cluster hypergraphs $H_{\text{red}}(\gamma), H_{\text{blue}}(\gamma), H_{\text{green}}(\gamma)$, then in the hypergraph H every remaining hyperedge in the ε-regular triple (V_i, V_j, V_k) is colored by one of s possible colors. As $e(V_i, V_j, V_k) \le (n/t)^3$, we can color these hyperedges in at most

$$s^{(n/t)^3} \tag{5.8}$$

ways. Let e_s be the number of triples $\{i, j, k\}$, $1 \le i < j < k \le t$, which are hyperedges in exactly s cluster hypergraphs. Hence, the number of 3-colorings, which yield the partition $V(H) = V_1 \dot\cup \cdots \dot\cup V_t$ of the vertex set $V(H)$ and the cluster hypergraphs $H_{\text{red}}(\gamma), H_{\text{blue}}(\gamma), H_{\text{green}}(\gamma)$, is by (5.6)–(5.8) with $e(V_i, V_j, V_k) \le (n/t)^3$, $1 \le i < j < k \le t$, at most

$$2^{h(6\gamma)n^3/6} \cdot 3^{\gamma n^3} \cdot \left(1^{e_1} 2^{e_2} 3^{e_3}\right)^{(n/t)^3} = 2^{h(6\gamma)n^3/6} \cdot 3^{\gamma n^3} \cdot \left(2^{e_2} 3^{e_3}\right)^{(n/t)^3}. \tag{5.9}$$

None of the cluster hypergraphs contains a Fano plane, and hence they have at most $e(B_t)$ hyperedges, i.e., $e(H_{\text{col}}(\gamma)) \le e(B_t) \le t^3/8$ for col $\in \{\text{red}, \text{blue}, \text{green}\}$. Observe that

$$\begin{aligned} 2e_2 + 3e_3 \le e_1 + 2e_2 + 3e_3 &= e(H_{\text{red}}(\gamma)) + e(H_{\text{blue}}(\gamma)) + e(H_{\text{green}}(\gamma)) \\ &\le 3e(B_t) \le \frac{3t^3}{8}, \end{aligned} \tag{5.10}$$

thus
$$e_2 \le \frac{3t^3}{16} - \frac{3e_3}{2}, \tag{5.11}$$
and we infer by using $2 < 3^{7/11}$ that
$$2^{e_2} \cdot 3^{e_3} \overset{(5.11)}{\le} 2^{3t^3/16-3e_3/2} \cdot 3^{e_3} < 3^{(7/11)(3t^3/16-3e_3/2)} \cdot 3^{e_3} \le 3^{21t^3/176+e_3/22}. \tag{5.12}$$

Assume that for every choice of a Fano plane-free coloring of the set of hyperedges of H we obtain
$$e_3 < \frac{t^3}{8} - 44\gamma t^3 - 22h(6\gamma)t^3.$$
Then, we have
$$2^{e_2} \cdot 3^{e_3} \overset{(5.12)}{<} 3^{t^3/8-2\gamma t^3-h(6\gamma)t^3}. \tag{5.13}$$

Recalling that there are at most T_0^n partitions of the vertex set V into at most T_0 classes and that there are at most $2^{3\binom{T_0}{3}} < 2^{T_0^3}$ choices for the cluster hypergraphs $H_{\text{red}}(\gamma), H_{\text{blue}}(\gamma), H_{\text{green}}(\gamma)$, we infer from (5.9) and (5.13) that the total number of such 3-colorings of H is at most
$$\begin{aligned} &T_0^n \cdot 2^{T_0^3} \cdot 2^{h(6\gamma)n^3/6} \cdot 3^{\gamma n^3} \cdot \left(3^{t^3/8-2\gamma t^3-h(6\gamma)t^3}\right)^{(n/t)^3} \\ &\qquad = T_0^n \cdot 2^{T_0^3} \cdot 2^{h(6\gamma)n^3/6} \cdot 3^{\gamma n^3} \cdot 3^{n^3/8-2\gamma n^3-h(6\gamma)n^3} \\ &\qquad\qquad < T_0^n \cdot 2^{T_0^3} \cdot 3^{n^3/8-\gamma n^3-5h(6\gamma)n^3/6} < 3^{e(B_n)} \end{aligned}$$
for sufficiently large n, which contradicts our assumption.

Hence, there exists a Fano plane-free 3-coloring of H, which yields a partition $V(H) = V_1\dot\cup\cdots\dot\cup V_t$, $t \le T_0$, and cluster hypergraphs $H_{\text{red}}(\gamma)$, $H_{\text{blue}}(\gamma)$, $H_{\text{green}}(\gamma)$ such that
$$e_3 \ge \frac{t^3}{8} - 44\gamma t^3 - 22h(6\gamma)t^3. \tag{5.14}$$
We infer
$$e_1 + e_2 \le e_1 + 2e_2 \overset{(5.10),(5.14)}{\le} 132\gamma t^3 + 66h(6\gamma)t^3. \tag{5.15}$$
Let H_3 be that hypergraph on the vertex set $[t]$, which consists of all hyperedges, which are contained in all three cluster hypergraphs. Let H' be the subhypergraph of H, which contains all those hyperedges from H, which correspond to the hyperedges in H_3, i.e., $\{i,j,k\} \in E(H_3)$ if and only if $E(V_i,V_j,V_k) \subseteq E(H')$.

Due to (5.14) and (5.1), by Theorem 4.1 there exists a partition $[t] = A\dot\cup B$ such that
$$e_{H_3}(A) + e_{H_3}(B) < \frac{\alpha}{2}t^3. \tag{5.16}$$

Set $X = \bigcup_{j\in A} V_j$ and $Y = \bigcup_{j\in B} V_j$. Then, it is

$$\begin{aligned} e_H(X) + e_H(Y) &\overset{(5.5)}{\leq} \gamma n^3 + (n/t)^3(e_{H_3}(A) + e_{H_3}(B) + e_1 + e_2) \\ &\overset{(5.15),(5.16)}{\leq} \gamma n^3 + (n/t)^3(\alpha t^3/2 + 132\gamma t^3 + 66h(6\gamma)t^3) \leq \gamma n^3 + \alpha n^3/2 + 132\gamma n^3 + 66h(6\gamma)n^3 \\ &\overset{(5.1)}{<} \alpha n^3, \end{aligned}$$

which yields the desired partition $V(H) = X\dot\cup Y$. $\square$

Alternatively, we could argue at the end of the proof of Theorem 5.1 (just right before inequality (5.14)) as follows. Let H' be the underlying hypergraph which consists of all hyperedges which are contained in one of the ε-regular 3-tuples of density at least γ in *every* color. Then H' must be an F-free hypergraph, as otherwise the corresponding 3-tuples that contain the hyperedges of the copy of F will form a copy of F in each of the cluster hypergraphs $H_{\text{red}}(\gamma), H_{\text{blue}}(\gamma), H_{\text{green}}(\gamma)$, and hence there will be many monochromatic copies of F in H which is impossible.

Note that

$$|E(H) \setminus E(H')| \overset{(5.5),(5.15)}{\leq} \gamma n^3 + (132\gamma t^3 + 66h(6\gamma)t^3)(n/t)^3 < \frac{\alpha}{2}n^3, \tag{5.17}$$

while on the other side, it holds

$$|E(H')| \overset{(5.14)}{\geq} \left(\frac{t^3}{8} - 44\gamma t^3 - 22h(6\gamma)t^3\right)(n/t)^3 \overset{(5.1)}{\geq} \mathrm{ex}(n,F) - \lambda(\alpha/2)n^3.$$

From above, it follows that B_n and H' are $(\alpha/2)$-close, and therefore, with (5.17), we immediately obtain that B_n and H are α-close. This, in turn implies the existence of bipartition of $V(H) = X\dot\cup Y$ such that $e(X) + e(Y) < \alpha n^3$.

5.1.3 Proof of Theorem 1.11

We prove the result only for $r = 3$, as the two-color-case is similar. (In fact, for $r = 2$ the first part of Theorem 1.11 already follows from (1.6).) Also note, the first claim of the theorem stating

$$c_{r,F}(n) \leq r^{\mathrm{ex}(n,F)+o(n^k)}$$

does not need any assumption on the s-stability of F. However, s-stability is only used in the last paragraph of Case 1, see below. Therefore we prove both claims simultaneously.

Given $\varepsilon > 0$, let $\omega > 0$ be given that satisfies the s-stability condition in Definition 1.10 for $\varepsilon/3$ and the hypergraph F. We choose positive ξ and ζ such that

$$4(\xi + \zeta) \leq \min\{\omega, \varepsilon/3\} \quad \text{and} \quad h(k!4\xi) + 4\xi \leq \frac{\pi_F \zeta}{k!2^{k-1}88}, \tag{5.18}$$

where $h(y) := -y\log y - (1-y)\log(1-y)$ for $0 < y < 1$ is the entropy function. Now apply counting lemma, Theorem 2.17, with F and $d_k = \xi$ obtaining $\delta_k > 0$. We may assume that

$$\delta_k \leq \xi/3, \tag{5.19}$$

as setting δ_k smaller makes the complexes we consider more regular (and therefore the statements still hold). We choose $\eta > 0$ as follows

$$\eta \leq 2\xi/3, \tag{5.20}$$

so that for every $a \geq 1/(2\eta)$, if a hypergraph on a vertices has at least

$$\pi_F(1 + \zeta/88)\binom{a}{k} \tag{5.21}$$

hyperedges, then it contains a copy of F. Note that because $\mathrm{ex}(n, F)/\binom{n}{k}$ is a monotone decreasing function, which converges to π_F, such choices are always possible. Recall also that $a_1 \geq (1/2\eta)$ for an equitable family of partitions (cf. property (a) of Definition 2.13). Next choose $\gamma > 0$ so that

$$\gamma \leq \min\{1/9, \zeta/88\}. \tag{5.22}$$

Let $\delta\colon \mathbb{N}^{k-1} \to (0, 1]$ and $f\colon \mathbb{N}^{k-1} \to \mathbb{N}$ be the functions guaranteed by Theorem 2.17. Also, we require, that the number of cliques of size k that are spanned by *any* $(\delta(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{d})$-regular $(n/a_1, k, k-1)$-complex should lie in the range

$$(1 \pm \gamma)\left(\frac{n}{a_1}\right)^k \Big/ \prod_{i=2}^{k-1} a_i^{\binom{k}{i}}.$$

The existence of an appropriately small function δ is asserted in [KRS02, Theorem 6.5]. The rôle of γ will become clear later in (5.25). Roughly speaking, after we regularize the hypergraph under consideration, we need good estimates on the number of hyperedges a polyad can contain. For this we apply some form of a counting lemma proven in [KRS02, Theorem 6.5] ("dense counting lemma") to an equitable family of partitions, in particular the last "layer" of this family forms a very regular partition of the $(k-1)$-subsets with precision δ. Also note, that choosing δ smaller does not affect the conclusion of Theorem 2.17.

Now, let m_0 be given by Theorem 2.17 and t_0 by Theorem 2.16. Further we choose n_0 larger than $t_0 \cdot m_0$ and another n_0 given again by Theorem 2.16.

Consider a hypergraph H on $n \geq n_0$ vertices with $c_{3,F}(H) \geq 3^{\mathrm{ex}(n,F)}$ (for the "furthermore"-part we consider $H \in \{H_1, \ldots, H_{s+1}\}$). We assume without loss of generality that $t_0!$ divides n, as otherwise we may delete less than $t_0!$ vertices and obtain a subhypergraph $H' \subset H$ with

$$c_{3,F}(H') \geq 3^{\mathrm{ex}(n,F) - O(n^{k-1})},$$

and it follows from the proof that the $O(n^{k-1})$ term does not harm us at all.

So fix any 3-hyperedge-coloring φ of H, with color classes $H_{\mathrm{green}}, H_{\mathrm{blue}}, H_{\mathrm{red}}$, without a monochromatic subhypergraph F. Apply Theorem 2.16 with the parameters k, $c = 3$, δ_k,

η and the functions f and δ specified above. We obtain from Theorem 2.16 an integer t_0 and a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ such that the properties specified in Theorem 2.16 hold. Roughly speaking, we know that H_{green}, H_{blue}, and H_{red} are $(\delta_k, *, f)$-regular with respect to the obtained family of partitions.

We discard from our consideration the following colored hyperedges in H.

- all hyperedges which are not in $\mathrm{C}ross_k(\mathscr{P}^{(1)})$, which are at most $\eta\binom{n}{k}$, and
- all hyperedges which are contained in $(\delta_k, *, f(\boldsymbol{a}^{\mathscr{P}}))$-irregular polyads with respect to one of the colors, hence at most
$$3\delta_k|V|^k = 3\delta_k n^k$$
such hyperedges, and
- furthermore, for every color we discard all hyperedges that are contained in $(\delta_k, *, f(\boldsymbol{a}^{\mathscr{P}}))$-regular polyads of density less than ξ, which are at most $3\xi\binom{n}{k}$.

So, in total we discard at most

$$\eta\binom{n}{k} + 3\delta_k n^k + 3\xi\binom{n}{k} \leq 4\xi n^k \tag{5.23}$$

hyperedges, where we used (5.19) and (5.20).

There are

$$N_p := \binom{a_1}{k} \cdot \prod_{i=2}^{k-1} a_i^{\binom{k}{i}} \tag{5.24}$$

$(k, k-1)$ polyads in the partition $\mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$. Due to the choice of δ, in every polyad $\boldsymbol{\mathcal{P}}(J)$ there are at most

$$E_p^+ := (1+\gamma)\left(\frac{n}{a_1}\right)^k \Big/ \prod_{i=2}^{k-1} a_i^{\binom{k}{i}} \tag{5.25}$$

many hyperedges in each of the three colors, red, blue and green, due to [KRS02, Theorem 6.5].

Let $p_{\text{green}}, p_{\text{blue}}, p_{\text{red}}$ denote the number of $(\delta_k, *, f(\boldsymbol{a}^{\mathscr{P}}))$-regular polyads of density at least ξ in the colors green, blue and red, respectively. We know that every "monochromatic" slice cannot have more than $\mathrm{ex}(a_1, F)$ such regular polyads, as otherwise, the counting lemma, Theorem 2.17, would imply that the hypergraph H contains a monochromatic copy of F which contradicts our choice of the coloring of the set of hyperedges of H.

Note that there are exactly

$$S := \prod_{i=2}^{k-1} a_i^{\binom{a_1}{i}}$$

different slices (see Section 2.4.5), while every polyad occurs in exactly

$$S \cdot \prod_{i=2}^{k-1} a_i^{-\binom{k}{i}}$$

many slices.

Thus, we infer by averaging for every color col $\in \{\text{green}, \text{blue}, \text{red}\}$ that the number p_{col} of polyads satisfies

$$p_{\text{col}} \leq \text{ex}(a_1, F) \cdot \prod_{i=2}^{k-1} a_i^{\binom{k}{i}}. \tag{5.26}$$

On the other hand, let e_j for $j \in [3]$ denote the number of $(\delta_k, *, f(\boldsymbol{a}^{\mathscr{P}}))$-regular polyads of density at least ξ in *exactly* j colors. We note the following simple identity

$$e_1 + 2e_2 + 3e_3 = p_{\text{green}} + p_{\text{red}} + p_{\text{blue}} \overset{(5.26)}{\leq} 3\text{ex}(a_1, F) \cdot \prod_{i=2}^{k-1} a_i^{\binom{k}{i}}. \tag{5.27}$$

Now we split our argument into two parts.

Case 1

Assume first that

$$e_3 \geq (1 - \zeta)\text{ex}(a_1, F) \cdot \prod_{i=2}^{k-1} a_i^{\binom{k}{i}}, \tag{5.28}$$

which means by (5.25) and (5.27) that the number of hyperedges contained in polyads which are regular in at most two colors is at most

$$3\zeta\text{ex}(a_1, F) \cdot \prod_{i=2}^{k-1} a_i^{\binom{k}{i}} \cdot E_p^+ \leq 3\zeta(1+\gamma) \cdot \frac{\text{ex}(a_1, F)}{a_1^k} \cdot n^k \overset{(5.22)}{\leq} 4\zeta \cdot \frac{\text{ex}(a_1, F)}{a_1^k} \cdot n^k \leq 4\zeta n^k.$$

We also discard these hyperedges. Hence, in view of (5.23), we discard at most $4(\zeta+\xi)n^k$ hyperedges in this case.

For a moment we ignore the different colors. We denote by H' the resulting hypergraph (consisting of the left-over hyperedges). The number of hyperedges in H' is at least $\text{ex}(n, F) - 4(\zeta + \xi)n^k \geq \pi_F\binom{n}{k} - \omega n^k$ (this follows trivially as $e(H) \geq \text{ex}(n, F)$, which is again implied by $c_{3,F}(n) \geq 3^{\text{ex}(n,F)}$). On the other hand, H' itself cannot have more than $\text{ex}(n, F)$ hyperedges. Otherwise, there would exist a copy F' of F in H'. This is however impossible as then Theorem 2.17 applies, which yields a copy of F even in every color. Indeed, the hyperedges of F' must lie in regular polyads of density at least ξ. Thus, conditions (i) and (ii) are fulfilled, and we therefore find a copy of F in H in *any* color, which is a contradiction. Thus, H' is F-free and $e(H') \leq \text{ex}(n, F)$.

From here, the first claim of the theorem follows immediately in view of the fact

$$e(H) \leq e(H') + 4(\zeta + \xi)n^k \overset{(5.18)}{<} \mathrm{ex}(n, F) + \varepsilon n^k.$$

As for the second claim, we argue as follows. If the inequality (5.28) holds for every $H \in \{H_1, \dots, H_{s+1}\}$, then for every $i \in [s+1]$ there exists $H_i' \subset H_i$, H_i' is F-free and $e(H_i') \geq \pi_F\binom{n}{k} - \omega n^k$. We infer from the s-stability that there exist $i, j \in [s+1], i \neq j$ such that H_i' and H_j' are $\varepsilon/3$-close. Also we surely have that $|H_i \Delta H_i'| \leq \varepsilon n^k/3$ and $|H_j \Delta H_j'| \leq \varepsilon n^k/3$, which implies that H_i and H_j are ε-close. This finishes the first case and we conclude the theorem in this case.

Case 2

Now, we argue that under the assumption $c_{3,F}(H) \geq 3^{\mathrm{ex}(n,F)}$ there always exists some coloring, for which inequality (5.28) holds. We then arrive at a contradiction assuming that this is not the case.

So assume that (5.28) does not hold for *any* 3-hyperedge coloring of H without a monochromatic copy of F. We can regularize the hypergraph H for every hyperedge coloring. Our goal is to show, that H cannot have too many hyperedge colorings.

We first bound the number of different $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable families of partitions which are t_0-bounded together with $(\delta_k, *, f(\boldsymbol{a}^{\mathscr{P}}))$-regular polyads in every color of density at least ξ, and due to the t_0-boundedness there are at most

$$\left(\prod_{i=1}^{k-1} t_0^{\binom{n}{i}}\right) \cdot 2^{3N_p} \leq t_0^{2n^{k-1}} \tag{5.29}$$

many of these. We also discarded at most $4\xi n^k$ (cf.(5.23)) many hyperedges from irregular and “sparse” polyads, over which we had no control, thus we upper bound the number of ways one can additionally choose and color these hyperedges by

$$\binom{\binom{n}{k}}{4\xi n^k} \cdot 3^{4\xi n^k} \leq 2^{h(k!4\xi)n^k} \cdot 3^{4\xi n^k}. \tag{5.30}$$

Now we are left to estimate the number of ways we can color the set of remaining hyperedges for some fixed family of partitions $\mathscr{P}$. There are at most

$$(1^{e_1} \cdot 2^{e_2} \cdot 3^{e_3})^{E_P^+} \tag{5.31}$$

many ways, where we consider all possible hyperedges a polyad can span and take into account in how many colors some particular polyad is regular.

Set $T_a := \prod_{i=2}^{k-1} a_i^{\binom{k}{i}}$. By assumption, $e_3 < (1-\zeta)\mathrm{ex}(a_1, F) \cdot T_a$ and hence by (5.27) we have

$$e_2 \leq \frac{3}{2}\Big(\mathrm{ex}(a_1, F) \cdot T_a - e_3\Big).$$

With $2 < 3^{7/11}$ we can upper bound (5.31) by

$$\left(2^{\frac{3}{2}(\mathrm{ex}(a_1,F)T_a - e_3)} \cdot 3^{e_3}\right)^{E_p^+} \leq \left(3^{(21/22)\mathrm{ex}(a_1,F)T_a + (1/22)e_3}\right)^{E_p^+}$$
$$\leq \left(3^{(21/22)\mathrm{ex}(a_1,F)T_a + (1/22)(1-\zeta)\mathrm{ex}(a_1,F)T_a}\right)^{E_p^+} = \left(3^{\mathrm{ex}(a_1,F)T_a - (1/22)\zeta\mathrm{ex}(a_1,F)T_a}\right)^{E_p^+}$$
$$\stackrel{(5.25)}{\leq} 3^{(1-\zeta/22)(1+\gamma)\mathrm{ex}(a_1,F)(n/a_1)^k}.$$

So, together with (5.29), (5.30) and (5.31), we upper bound the number $c_{3,F}(\mathcal{H})$ by

$$t_0^{2n^{k-1}} \cdot 2^{h(k!4\xi)n^k} \cdot 3^{4\xi n^k} \cdot 3^{(1-\zeta/22)(1+\gamma)\mathrm{ex}(a_1,F)(n/a_1)^k}$$
$$\stackrel{(5.21)}{\leq} t_0^{2n^{k-1}} 3^{h(k!4\xi)n^k + 4\xi n^k + (1-\zeta/22)(1+\gamma)(1+\zeta/88)\pi_F\binom{a_1}{k}(n/a_1)^k}$$
$$\stackrel{(5.18),(5.22)}{\leq} 3^{\mathrm{ex}(n,F) - \zeta\pi_F n^k/(k!88)},$$

and this contradicts the assumptions of the theorem and finishes the proof, because we have shown that Case 2 never occurs, and therefore always Case 1 applies. □

We note that, in fact, we proved here a slightly stronger result which reads as follows:

Theorem 5.2. *Let $k, s \in \mathbb{N}$, $k \geq 2$ and $r = 2$ or 3. Let F be a k-uniform hypergraph, such that $\pi_F > 0$. Furthermore suppose that F is s-stable. Then, for every $\varepsilon > 0$ there exist $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that the following holds.*

Among any $s+1$ many k-uniform hypergraphs $H_1, \dots, H_{s+1}$ on $n \geq n_0$ vertices that satisfy $c_{r,F}(H_i) \geq r^{\mathrm{ex}(n,F)-\alpha n^k}$ for every $i \in [s+1]$, there exist two which are ε-close.

Remark 5.3. Recall that when F is a graph, it is 1-stable [Sim68]. The above theorem looks slightly stronger than the one in [ABKS04, Lemma 2.1] (due to the allowed lower bound $2^{\mathrm{ex}(n,F)-o(n^2)}$ instead of $2^{\mathrm{ex}(n,F)}$). However, this "better" bound $2^{\mathrm{ex}(n,F)-o(n^2)}$ is implicit in [ABKS04], this was noted by the authors in [LPS] and by Pikhurko [Pik09].

5.2 Exact results for some hypergraphs

5.2.1 Fano plane

Proof of Theorem 1.12. We prove only the case $r = 3$, as the proof for two colors is similar. We first fix all constants needed for the proof. Let ξ, ϱ, and ζ be defined by the following equations

$$3^6 - 1 = 3^{6-\xi}, \quad 3^4 - 1 = 3^{4-\varrho}, \quad \text{and} \quad (3h(2\zeta)+1)(1+8\zeta)\log_3(2) = 1-\zeta\,, \quad (5.32)$$

where $h(x)$ is the entropy function. Recall that $h(x) \to 0$ as $x \to 0$ and, since $\log_3(2) < 1$, there exists such a $\zeta > 0$ satisfying the above such that $(3h(2\gamma)+1)(1+8\gamma)\log_3(2) < 1-\gamma$

for all $0 < \gamma < \zeta$. We set

$$\gamma := \min\left\{\frac{\xi}{2000}, \frac{\zeta}{2}\right\} \le \frac{1}{25} \qquad \text{and} \qquad \delta := \frac{\varrho\gamma^3}{1000}\,, \tag{5.33}$$

For the main steps of the proof it is sufficient to keep in mind that

$$0 < \delta \ll \gamma \ll \varrho, \xi, \zeta.$$

Let $n_0 = n_0(3, \delta)$ be given by Theorem 1.11 (applied with $\varepsilon := \delta$), or alternatively by Theorem 5.1, and set $n_r = n_3 \ge n_0 + \binom{n_0}{3}$ sufficiently large.

The proof is similar to that in [ABKS04] and proceeds by contradiction. Assume that we are given a hypergraph H on $n > n_3$ vertices with $c_{3,F}(H) \ge 3^{e(B_n)+m}$ for some $m \ge 0$. We show the following claim.

Claim 5.4. *If $c_{3,F}(H) \ge 3^{e(B_n)+m}$ for some $m \ge 0$ and H is not the balanced, complete, bipartite hypergraph B_n, then there exists an induced subhypergraph H' on n' vertices with $n' \ge n - 3$ and $c_{3,F}(H') \ge 3^{e(B_{n'})+m+1}$.*

Inductively, we arrive at some subhypergraph H_0 with at least n_0 vertices that admits at least $3^{e(B_{n_0})+\binom{n_0}{3}+1}$ Fano plane-free 3-colorings of the set of hyperedges, which is impossible and yields the desired contradiction and it is left to verify Claim 5.4. □

Proof of Claim 5.4. Let H be a hypergraph on n vertices, $H \ne B_n$ and $c_{3,F}(H) \ge 3^{e(B_n)+m}$ with $m \ge 0$. Clearly, this implies $e(H) \ge e(B_n)$. Without loss of generality we may assume that $\delta(H) \ge \delta(B_n)$. Otherwise let v be a vertex of minimum degree in H and consider $H' := H - v$. Since $e(B_{n-1}) = e(B_n) - \delta(B_n) \le e(B_n) - \delta(H) - 1$ we have

$$c_{3,F}(H') \ge \frac{c_{3,F}(H)}{3^{\delta(H)}} = 3^{e(B_n)-\delta(H)+m} \ge 3^{e(B_{n-1})+m+1}.$$

In view of (4.2), from now on we may assume $\delta(H) \ge \delta(B_n) \ge 3n^2/8 - n$. Consider a partition of $V(H) = X \dot\cup Y$, which minimizes $e(X) + e(Y)$. Because of Theorem 5.1 we know that $e(X) + e(Y) < \delta n^3$ and, hence

$$e(H) < e(B_n) + \delta n^3$$

and it follows from $e(H) \ge e(B_n)$ that

$$e(X, Y) \ge e(B_n) - \delta n^3 \ge n^3/8 - n^2 - \delta n^3\,,$$

which in turn implies

$$n/2 - 2\sqrt{\delta}n \le |X|, |Y| \le n/2 + 2\sqrt{\delta}n. \tag{5.34}$$

Our argument splits into two cases depending on the link(graph). Recall, that for a vertex v of H its link is $L(v) := \{\{u, w\} \colon \{v, u, w\} \in E(H)\}$, which is a graph on $V(H)$.

First (in Case 1) we will assume that there exists a vertex v with at least γn^2 link edges in its "own" partition class.

Case 1 (H has the property that $\exists Z \in \{X,Y\}$ and $\exists v \in Z\colon\ |L(v)\cap[Z]^2| \geq \gamma n^2$). Without loss of generality we may assume $v \in Y$ with $|L(v)\cap[Y]^2| \geq \gamma n^2$. The minimality of $e(X)+e(Y)$ implies, that $|L(v)\cap[X]^2| \geq \gamma n^2$, as otherwise we could move v to X decreasing $e(X)+e(Y)$.

We split the Fano plane-free colorings of H into two classes $\mathcal{C}_1$ and $\mathcal{C}_2 = \overline{\mathcal{C}_1}$. Let $\mathcal{C}_1$ be the set of those colorings for which there exist $L'_Y \subset L(v)\cap[Y]^2$ and $L'_X \subset L(v)\cap[X]^2$, of size at least $\gamma n^2/4$ each, and all hyperedges of the form $\{v\}\cup f$ with $f \in L'_X \cup L'_Y$ have the same color.

For a fixed coloring from $\mathcal{C}_1$ there exist matchings $M_X \subset L'_X$ and $M_Y \subset L'_Y$, and $\min\{|M_X|,|M_Y|\} \geq \gamma n/5$. For three link edges f_1, f_2, f_3 with $f_1 \in M_Y$ and $f_2, f_3 \in M_X$ let $t_1,t_2,t_3,t_4 \in [V]^3$ be four triples (not necessarily hyperedges of H) such that $\{\{v\}\cup f_i\colon\ i=1,2,3\}\cup\{t_1,\dots,t_4\}$ forms a Fano plane. Note that each of the triples t_1,t_2,t_3,t_4 contains precisely one vertex from $f_1 \subset Y$ and precisely one vertex from each of f_2 and $f_3 \subset X$. (In fact, there are two different sets of four triples $t_1,\dots,t_4$ for any given f_1,f_2,f_3 and we just fix one of those two sets.) Since $\{v\}\cup f_i$ are of the same color either one of the triples t_j must be missing in H or there are only 3^4-1 ways to color t_1,t_2,t_3,t_4. Since $|M_X|,|M_Y| \geq \gamma n/5$ there are at least $\frac{\gamma n}{5}\binom{\gamma n/5}{2}$ possible choices for f_1,f_2,f_3 and since there are at most $\delta n^3 \leq \gamma^3 n^3/1000$ hyperedges absent between X and Y, there are at least $\gamma^3 n^3/500$ such Fano planes present in H for a fixed coloring in $\mathcal{C}_1$. Furthermore, note that for two different choices of f_1,f_2,f_3 and f'_1,f'_2,f'_3 the corresponding sets $\{t_1,\dots,t_4\}$ and $\{t'_1,\dots,t'_4\}$ are disjoint. Hence we obtain the following estimate on $|\mathcal{C}_1|$

$$\begin{aligned}
|\mathcal{C}_1| &\leq 3\binom{\binom{|X|}{2}}{\gamma n^2/4}\binom{\binom{|Y|}{2}}{\gamma n^2/4}\frac{3^{e(H)}}{3^{4\gamma^3 n^3/500}}(3^4-1)^{\gamma^3 n^3/500}\\
&\overset{(5.32)}{\leq} 3\cdot 2^{n^2}\cdot 3^{e(B_n)+\delta n^3-4\gamma^3 n^3/500+(4-\varrho)\gamma^3 n^3/500}\\
&\overset{(5.33)}{=} 3\cdot 2^{n^2}\cdot 3^{e(B_n)-\delta n^3}\,.
\end{aligned}$$

Consequently, for large enough n we have

$$|\mathcal{C}_1| \leq 3^{e(B_n)-1}.$$

Let $\mathcal{C}_2$ be the Fano plane-free edge colorings of H which do not belong to $\mathcal{C}_1$, i.e., the family of those colorings for which there does not exist $L'_Y \subset L(v)\cap[Y]^2$ and $L'_X \subset L(v)\cap[X]^2$, of size at least $\gamma n^2/4$ each, and such that all hyperedges of the form $\{v\}\cup f$ with $f \in L'_X \cup L'_Y$ have the same color. We have just shown that

$$|\mathcal{C}_2| \geq 3^{e(B_n)+m-1}\,.$$

Next we estimate the number of colorings of the set of hyperedges incident to v, which can be extended to a coloring in $\mathcal{C}_2$. For a set $W \subseteq V(H)$ we say $e \in E(H)$ *is a hyperedge* *from v to W* if $v \in e$ and $(e \setminus \{v\}) \subset W$.

For any coloring from $\mathcal{C}_2$, by definition, for every col $\in$ {red, blue, green} there is a vertex class $V_{\mathrm{col}} \in \{X, Y\}$ such that there are at most $\gamma n^2/4$ hyperedges from v to V_{col}, since otherwise the coloring would belong to $\mathcal{C}_1$. Note that because of (5.33) and (5.34) the size of $[V_{\mathrm{col}}]^2$ is at most $n^2/8 + \gamma n^2$ and, consequently, there are at most

$$\binom{n^2/8+\gamma n^2}{\gamma n^2/4} \le 2^{h\left(\frac{2\gamma}{1+8\gamma}\right)(1+8\gamma)n^2/8} \overset{(5.33)}{\le} 2^{h(2\gamma)(1+8\gamma)n^2/8}$$

ways to choose the hyperedges of color col between v and V_{col}.

Since $|L(v) \cap [X]^2|, |L(v) \cap [Y]^2| \ge \gamma n^2$ it is impossible that $V_{\mathrm{red}} = V_{\mathrm{blue}} = V_{\mathrm{green}}$. Hence for two colors, say red and blue, there will be at most $\gamma n^2/4$ hyperedges from v to, say, $X = V_{\mathrm{red}} = V_{\mathrm{blue}}$ (the case $Y = V_{\mathrm{red}} = V_{\mathrm{blue}}$ is symmetric here and the analysis is independent from the earlier assumption $v \in Y$). Then for the remaining third color there will be at most $\gamma n^2/4$ hyperedges of color green from v to $Y = V_{\mathrm{green}}$. Now we can color the remaining hyperedges from v to X only green, and we can color the remaining hyperedges (there are at most $n^2/8 + \gamma n^2$) from v to Y with two colors, red and blue. We also had only 6 different possibilities to choose $V_{\mathrm{red}}, V_{\mathrm{blue}}, V_{\mathrm{green}} \in \{X, Y\}$ in such a way.

Finally, there are at most $n^2/4$ hyperedges, that contain v and intersect both X and Y, and they can be colored arbitrarily, so in total in at most $3^{n^2/4}$ ways. Summarizing the above, we can estimate the number of possible colorings of the hyperedges incident with v (which extend to a coloring in $\mathcal{C}_2$) from above by

$$\begin{aligned} 6 \cdot 2^{3h(2\gamma)(1+8\gamma)n^2/8} \cdot 2^{(1+8\gamma)n^2/8} \cdot 3^{n^2/4} &= 6 \cdot 3^{(3h(2\gamma)+1)(1+8\gamma)\log_3(2)n^2/8+n^2/4} \\ &\overset{(5.32)}{\le} 3^{2+(1-\gamma)n^2/8+n^2/4} = 3^{3n^2/8-\gamma n^2/8+2} \overset{(4.2)}{\le} 3^{\delta(B_n)-2}. \end{aligned}$$

Setting $H' := H - v$ we obtain

$$c_{3,F}(H') \ge \frac{|\mathcal{C}_2|}{3^{\delta(B_n)-2}} \ge \frac{3^{e(B_n)+m-1}}{3^{\delta(B_n)-2}} = 3^{e(B_{n-1})+m+1},$$

which proves Claim 5.4 for hypergraphs H satisfying the assumptions of Case 1.

Next we consider the case that every vertex v has at most γn^2 link edges in its own partition class.

Case 2 (H has the property that $\forall Z \in \{X, Y\}$ and $\forall v \in Z$: $|L(v) \cap [Z]^2| \le \gamma n^2$). As still $H \ne B_n$ there exists (without loss of generality) a hyperedge $e = \{v_1, v_2, v_3\} \subset Y$. Let $L := \bigcap_{i=1}^{3} L(v_i) \cap [X]^2$. From $\delta(H) \ge \delta(B_n) \ge 3n^2/8 - n$ it follows that $|L| \ge (1-4\gamma)\binom{|X|}{2} > (2/3 + 1/6)\binom{|X|}{2}$ (see (5.33)). By Turán's theorem [Tur41] and (5.34) we find at least $\frac{1}{36}\binom{|X|}{2} \ge \frac{1}{360}n^2$ edge-disjoint K_4's in L. Denote them by $K^1, \dots, K^q$, where

$$q \ge \frac{1}{360}n^2. \tag{5.35}$$

Since $K^j \subset L$ for every $j = 1, \dots, q$, every such K^j forms together with the hyperedge e a Fano plane. Fixing a color for e we can color the 6 hyperedges that correspond to the edges of every K^j in only $3^6 - 1$ instead of 3^6 different ways.

Set $H' := H - \{v_1, v_2, v_3\}$. Let E_e denote the set of hyperedges of H which contain at least one vertex from $e = \{v_1, v_2, v_3\}$. Obviously, $|E_e| \le 3\gamma n^2 + 3\binom{|X|}{2} + 3|X||Y|$. It follows from the choice of $\delta \ll \gamma$ (see (5.33)), $e(X) + e(Y) < \delta n^3$, and $e(H) \ge e(B_n)$, that

$$\begin{aligned} |E_e| \overset{(5.34)}{\le} \frac{9}{8}n^2 + 4\gamma n^2 \overset{(4.2)}{\le} \delta(B_n) + \delta(B_{n-1}) + \delta(B_{n-2}) + 5\gamma n^2 \\ = e(B_n) - e(B_{n-3}) + 5\gamma n^2 \,. \end{aligned}$$

We can color the set of hyperedges of E_e in at most

$$\frac{3^{|E_e|}}{3^{6q}}(3^6 - 1)^q \overset{(5.32)}{=} 3^{|E_e| - \xi q}$$

ways. Consequently,

$$c_{3,F}(H') \ge 3^{e(B_n)+m-|E_e|+\xi q} \ge 3^{e(B_{n-3})+m-5\gamma n^2+\xi q} \overset{(5.33),(5.35)}{\ge} 3^{e(B_{n-3})+m+1} ,$$

which concludes Case 2 and finishes the proof of Claim 5.4. □

5.2.2 Further notation

For a partition $\mathcal{P}$ of the vertex set $V(H)$, i.e., $V(H) = V_1 \dot\cup \cdots \dot\cup V_\ell$, a hyperedge e is called *crossing* if e intersects each class V_i, $i \in [\ell]$, in at most one vertex. Let $E_{\text{cross}}(\mathcal{P})$ be the set of all crossing hyperedges in the hypergraph H with respect to the partition $\mathcal{P}$. Moreover, let $E_{\text{noncross}}(\mathcal{P}) := E(H) \setminus E_{\text{cross}}(\mathcal{P})$ be the set of all *non-crossing* hyperedges in H, consisting of all hyperedges $e \in E(H)$, which intersect class V_i for some $i \in [\ell]$ in at least 2 vertices, and set $e_{\text{noncross}}(\mathcal{P}) := |E_{\text{noncross}}(\mathcal{P})|$. More generally, for k mutually disjoint subsets $U_1, \dots, U_k$ of V we denote by $E_H(U_1, \dots, U_k)$ the set of all hyperedges in H that intersect every subset U_i, $i \in [k]$, in exactly one vertex, and its cardinality is denoted by $e(U_1, \dots, U_k) := |E_H(U_1, \dots, U_k)|$.

For $t \in [k]$, and t pairwise distinct vertices $v_1, \dots, v_t$ let $L_H(v_1, \dots, v_t)$ be the set of all $(k-t)$-element subsets $S \subseteq V(H)$, such that $v_1, \dots, v_t$ together with S form a hyperedge in the k-uniform hypergraph H, i.e.,

$$L_H(v_1, \dots, v_t) = \{e \setminus \{v_1, \dots, v_t\} : e \in E(H) \text{ and } v_1, \dots, v_t \in e\}.$$

We call $L_H(v_1, \dots, v_t)$ the $(k-t)$-uniform *common link hypergraph*, or *common link graph* if $k - t = 2$.

We often will be interested in how hyperedges in the link of a vertex intersect a particular partition. For a k-uniform hypergraph $H = (V, E)$, a partition $\mathcal{P}$ of its vertex set with $V(H) = V_1 \dot\cup \cdots \dot\cup V_\ell$ into ℓ mutually disjoint classes, and any vertex $v \in V$, we distinguish between three different types of hyperedges incident to v. Let $v \in V_j$ be a vertex for some $j \in [\ell]$. We refer to those hyperedges $e \in E(H)$ incident to vertex v and intersecting every class $V_i, i \in [\ell]$, in at most one vertex as *crossing* hyperedges. Moreover, hyperedges incident to vertex v, that intersect class V_j in exactly one further vertex different from v and else intersect any other class $V_i, i \in [\ell] \setminus \{j\}$, in at most one vertex are referred to as *defective* hyperedges. Finally, all remaining hyperedges incident to vertex v are called *bad* hyperedges. More formally, crossing hyperedges incident to vertex v form the following subset of $E(H)$:

$$E_{\text{cross}}(v) := \{e \in E(H)\colon\ v \in e \text{ and } \forall i \in [\ell] : |e \cap V_i| \leq 1\},$$

while the set of defective hyperedges incident to vertex $v \in V_j$ is

$$E_{\text{defect}}(v) := \{e \in E(H)\colon\ v \in e \text{ and } |e \cap V_j| = 2 \text{ and } \forall i \in [\ell] \setminus j : |e \cap V_i| \leq 1\,\}.$$

Let $E_{\text{bad}}(v) = \{e \in E(H)\colon\ v \in e\} \setminus (E_{\text{cross}}(v) \dot\cup E_{\text{defect}}(v))$ be the set of bad hyperedges, or, equivalently, for $v \in V_j$:

$$E_{\text{bad}}(v) := \{e \in E(H)\colon\ v \in e \text{ and, } |e \cap V_j| \geq 3 \text{ or } \exists i \in [\ell] \setminus \{j\} \text{ with } |e \cap V_i| \geq 2\}.$$

Let $\tau\colon [\ell] \longrightarrow \{0, 1, \ldots, k\}$ be a function such that $\sum_{i=1}^{\ell} \tau(i) = k$. Then, for a k-element subset (hyperedge) e of V we say that e is of type τ, if $|e \cap V_i| = \tau(i)$ for each $i \in [\ell]$. We thus may formulate different types of hyperedges by specifying the types of these hyperedges. Therefore, for example, a crossing hyperedge has type τ, where k elements of $[\ell]$ are mapped to 1 and the remaining $(\ell - k)$ are mapped to 0. Note that there are $\binom{k+\ell-1}{\ell-1}$ distinct types of hyperedges with respect to the partition $V(H) = V_1 \dot\cup \cdots \dot\cup V_\ell$, provided $V_i \neq \varnothing$ for each $i \in [\ell]$. We also occasionally say a type τ intersects the class V_i if $\tau(i) \geq 1$.

For a vertex $v \in V(H)$ and a type τ associated with v we write $\deg^{\tau}(v) := |E^{\tau}(v)|$, where $E^{\tau}(v)$ denotes the set of all those hyperedges of type τ in H, which are incident to vertex v.

We also speak about crossing, defective or bad types, when we consider types of crossing, defective or bad hyperedges, respectively. Note however, a hyperedge might be of bad or defective type depending on the vertex under consideration incident to it.

5.2.3 Generalized triangles T_3 and T_4

Recall that T_k denotes the k-uniform generalized triangle. Here we prove Theorem 1.13, namely, for $k = 3$ or $k = 4$, and $r = 2$ or $r = 3$, and n sufficiently large it is

$$c_{r,T_k}(n) = r^{\text{ex}(n,T_k)}.$$

However, due to the similarity of the arguments, we only give a proof in the case of $r = 3$ colors and the 4-uniform generalized triangle T_4. Recall that for T_4 and n sufficiently large, the extremal hypergraph on n vertices is the balanced, complete, 4-partite, 4-

uniform Turán hypergraph $\mathcal{T}_4^{(4)}(n)$.

Proof of Theorem 1.13. Here we only prove the case $r = 3$ and $k = 4$.

Let n_0 be given by Theorem 1.11 (applied with δ as ε), we will specify δ below in (5.41) and let $n_{r,k} = n_{3,4} \geq n_0$ be sufficiently large.

The proof is similar to that in [LPRS09] and proceeds by contradiction. Assume that we are given a hypergraph H on $n > n_{3,4}$ vertices with $c_{3,T_4}(H) \geq 3^{\mathrm{ex}(n,T_4)+m}$ for some $m \geq 0$. We show the following claim.

Claim 5.5. *If $c_{3,T_4}(H) \geq 3^{\mathrm{ex}(n,T_4)+m}$ for some $m \geq 0$ and H is not the Turán hypergraph $\mathcal{T}_4^{(4)}(n)$, then there exists an induced subhypergraph H' on n' vertices with $n' \geq n-2$ and*

$$c_{3,T_4}(H') \geq 3^{\mathrm{ex}(n',T_4)+m+1}. \tag{5.36}$$

Using Claim 5.5 (notice that $H' \neq \mathcal{T}_4^{(4)}(n')$), inductively, we arrive at some subhypergraph H_0 of H on at most n_0 vertices which admits at least $3^{\mathrm{ex}(n_0,T_4)+\binom{n_0}{4}+1}$ monochromatic T_4-free 3-colorings of the set of hyperedges, which is impossible and yields the desired contradiction. This proves Theorem 1.13, and thus, it is left to verify Claim 5.5. $\square$

Proof of Claim 5.5. Let H be a hypergraph on n vertices, $H \neq \mathcal{T}_4^{(4)}(n)$ and let

$$c_{3,T_4}(H) \geq 3^{\mathrm{ex}(n,T_4)+m}$$

with $m \geq 0$. Clearly, this implies $e(H) \geq \mathrm{ex}(n,T_4)$.

Without loss of generality we may assume that the minimum degree of H satisfies

$$\delta(H) \geq \delta(\mathcal{T}_4^{(4)}(n)) \geq \left\lfloor \frac{n}{4} \right\rfloor^3. \tag{5.37}$$

Otherwise, let v be a vertex of minimum degree in H and consider the subhypergraph $H' := H - \{v\}$. Since $\mathrm{ex}(n-1,T_4) = \mathrm{ex}(n,T_4) - \delta(\mathcal{T}_4^{(4)}(n)) \leq \mathrm{ex}(n,T_4) - (\delta(H)+1)$ we have

$$c_{3,T_4}(H') \geq \frac{c_{3,F}(H)}{3^{\delta(H)}} \geq 3^{\mathrm{ex}(n,T_4)-\delta(H)+m} \geq 3^{\mathrm{ex}(n-1,T_4)+m+1}, \tag{5.38}$$

which yields already (5.5). Consequently, from now on we may assume

$$\delta(H) \geq \delta(\mathcal{T}_4^{(4)}(n)) \geq \lfloor n/4 \rfloor^3.$$

Consider a partition $\mathcal{P}$ with $V(H) = V_1 \dot\cup V_2 \dot\cup V_3 \dot\cup V_4$, which, among all partitions of $V(H)$ into four nonempty classes, maximizes $e_H(V_1,V_2,V_3,V_4)$, and therefore minimizes $e(H) - e_H(V_1,V_2,V_3,V_4)$. Since the generalized triangle T_4 is 1-stable, as proved by Pikhurko [Pik08], by Theorem 1.11 we know that for our choice of $\delta > 0$, H and $\mathcal{T}_4^{(4)}(n)$ are δ-close and we therefore have

$$e(H) - e_H(V_1,V_2,V_3,V_4) < \delta n^4, \tag{5.39}$$

which with (1.8) gives the upper bound $e(H) \leq \lceil n/4 \rceil^4 + \delta n^4$ on the number of hyperedges in H. For $0 < \delta \leq 1/4^5$ with $e(H) \geq \mathrm{ex}(n, T_4) \geq \lfloor n/4 \rfloor^4$, hence $e_H(V_1, V_2, V_3, V_4) \geq \lfloor n/4 \rfloor^4 - \delta n^4$, we obtain the following lower and upper bounds on the sizes of the classes V_i for all $i \in [4]$:

$$n/4 - 3\delta^{1/4} n \leq |V_i| \leq n/4 + 3\delta^{1/4} n. \tag{5.40}$$

To see this, let $|V_i| = n/4 + 3pn/4$ for some $i \in [4]$ and $p \geq 0$. Then, as the product of three positive numbers with given sum is maximal if all are the same, we must have (neglecting the roundings)

$$\begin{aligned} &\left(\frac{n}{4} + \frac{3p}{4} n\right) \cdot \left(\frac{n}{4} - \frac{p}{4} n\right)^3 \geq \left(\frac{n}{4}\right)^4 - \delta n^4 \\ \Longleftrightarrow &(1+3p) \cdot (1-p)^3 \geq 1 - 4^4 \delta \\ \Longrightarrow &1 - 3p^4 \geq 1 - 4^4 \delta \qquad\qquad \text{for } p \leq 3/4 \\ \Longrightarrow &4\delta^{1/4} \geq p, \end{aligned}$$

hence $|V_i| \leq n/4 + 3\delta^{1/4} n$. Moreover, since $(1+3p)(1-p)^3$ is decreasing for $p \geq 0$, it is not possible that $p \geq 3/4$, as

$$(1+3p) \cdot (1-p)^3 \leq \frac{13}{4} \cdot \left(\frac{1}{4}\right)^3 = \frac{13}{4^4} < 1 - 4^4 \delta$$

for $0 < \delta \leq 1/4^5$.

On the other hand, if $|V_i| = n/4 - 3pn/4$ for some $i \in [4]$ and $p \geq 0$, then as above we must have

$$\begin{aligned} &\left(\frac{n}{4} - \frac{3p}{4} n\right) \cdot \left(\frac{n}{4} + \frac{p}{4} n\right)^3 \geq \left(\frac{n}{4}\right)^4 - \delta n^4 \\ \Longleftrightarrow &(1-3p) \cdot (1+p)^3 \geq 1 - 4^4 \delta \\ \Longrightarrow &1 - 3p^4 \geq 1 - 4^4 \delta \\ \Longrightarrow &4\delta^{1/4} \geq p, \end{aligned}$$

hence $|V_i| \geq n/4 - 3\delta^{1/4} n$.

Now our argument splits into three cases depending on the link hypergraph of a vertex. First we assume that there exists a vertex v incident to at least βn^3 *bad* hyperedges with respect to the partition $\mathcal{P}$ of the vertex set $V(H)$ (Case 1). If this is not the case, then we assume that there exists a vertex v, which is incident to at least βn^3 *defective* hyperedges with respect to the partition $\mathcal{P}$ (Case 2). Finally, if neither Case 1 nor Case 2 holds, we deal with Case 3, where every vertex is adjacent to at most $2\beta n^3$ many *defective* or *bad* hyperedges. Thus, by assumption (5.37) on the minimum degree, since we choose $0 < \beta \ll 1$ we know that every vertex is adjacent mostly to *crossing* hyperedges with respect to the partition $\mathcal{P}$.

For that we set $\beta, \delta > 0$ as follows:

$$\beta \leq \frac{7}{11} \cdot \left(\frac{1}{32}\right)^3 \quad \text{and} \quad h(\beta/12) \leq \frac{1}{9 \cdot 32^3}, \quad \text{and} \quad \delta \leq \min\left\{(\beta/10)^4, \frac{1}{4^{24}}\right\}. \tag{5.41}$$

However, it is sufficient to keep in mind that

$$0 < \delta \ll \beta \ll 1. \tag{5.42}$$

Case 1 (H satisfies $\exists i \in \{1,2,3,4\}$ and $\exists v \in V_i$: $|E_{bad}(v)| \geq \beta n^3$). Assume without loss of generality that $i = 1$. Let $v \in V_1$ be a vertex such that $|E_{bad}(v)| \geq \beta n^3$. Note that there are 16 types of *bad* hyperedges incident to vertex v. Thus, for at least one type τ we know $|E^{\tau}(v)| \geq \beta n^3/16$. Therefore, there exists another vertex $w \neq v$ such that the common link graph $L_H(v,w)$ contains at least $\beta n^2/16$ edges, which are contained in some class V_j for some $j \in [4]$, that is, together with any edge from the link graph $L_H(v,w)$, the vertices v and w form a hyperedge of type τ. Then we may find greedily a matching $M \subseteq [V_j]^2$ and $M \subseteq L_H(v,w)$ of size at least $\beta n/9$. Note here, that an already constructed matching of size x can be extended as long as $2x(n/4 + 3\delta^{1/4}n) < \beta n^2/16$, hence we obtain a matching of size at least $\beta n/9$ for $0 < \delta \leq (1/97)^4$.

Now consider an edge $\{x,y\}$ from the matching M. For each $i \in [4] \setminus \{j\}$ we may take each time one vertex v_i from every class $V_i \setminus \{v,w\}$. Let these vertices be v_1, v_2, v_3. Then, for each such choice of v_1, v_2, v_3 these form together with vertex x or y a 4-element set. Moreover, adding the existing hyperedge $\{x,y,v,w\} \in E$, we obtain a copy of T_4, which is a subhypergraph of H unless $\{x, v_1, v_2, v_3\}$ or $\{y, v_1, v_2, v_3\}$ is missing, i.e., is not a hyperedge in H. For $\delta \leq (1/96)^4$ and n sufficiently large, there are at least $(n/4 - 3\delta^{1/4}n - 2)^3 \geq n^3/100$ possibilities to choose such a triple (v_1, v_2, v_3). Moreover, we may do this for any of the at least $\beta n/9$ edges in M, each time obtaining distinct pairs of 4-tuples $\{x, v_1, v_2, v_3\}$ and $\{y, v_1, v_2, v_3\}$, as each time we take another matching edge $\{x,y\}$. Since at most δn^4 hyperedges $\{x, v_1, v_2, v_3\}$ or $\{y, v_1, v_2, v_3\}$ are missing in H, we find for $0 < \delta \leq \beta/9000$, which holds by (5.41), at least

$$(n^3/100)(\beta n/9) - \delta n^4 \geq \beta n^4/1000 \tag{5.43}$$

copies of T_4, which are subhypergraphs in H.

Now, let F_1 and F_2 be such distinct subhypergraphs T_4. Since M is a matching, by our considerations from above we know that F_1 and F_2 either are hyperedge-disjoint, or they share a single hyperedge that consists of the vertices v, w and a certain edge e from the matching M. This hyperedge corresponds to the "third" hyperedge in the definition of the generalized triangle T_4, i.e., this hyperedge contains the symmetric difference of the first two. However, the point is that once the color of the hyperedge $\{v,w\} \cup e$ is fixed, we can color the two remaining hyperedges in each subhypergraph T_4 found in the described way in at most 8 instead of 9 ways, to exclude a monochromatic T_4. Applying the same considerations to all matching edges $e \in M$ with the corresponding subhypergraphs T_4, we obtain the following possibilities for coloring the set of hyperedges of H:

- for every matching edge $e \in M$ the hyperedge $e \cup \{v,w\}$ may be colored in at most

3 ways,

- by (5.43) there exist at least $\beta n^4/1000$ pairwise distinct subhypergraphs T_4, and hence at least $2\beta n^4/1000 = \beta n^4/500$ distinct hyperedges of H, such that two hyperedges of a single subhypergraph T_4 may be colored in at most 8 instead of 9 ways,
- finally, the set of remaining hyperedges may be colored arbitrarily by at most 3 colors.

This way, for $0 < \delta \leq \beta/10^4$, which holds by (5.41), and n sufficiently large, with (5.39) we bound the number of 3-colorings of the set of hyperedges of H from above by

$$3^{\mathrm{ex}(n,T_4)+\delta n^4-\beta n^4/500} \cdot 8^{\beta n^4/1000} \ll 3^{\mathrm{ex}(n,T_4)}, \tag{5.44}$$

which contradicts the assumption $c_{3,T_4}(H) \geq 3^{\mathrm{ex}(n,T_4)}$.

Therefore, we have shown that Case 1 never holds, which we assume in the following.

Case 2 (H satisfies $\exists i \in \{1,2,3,4\}$ and $\exists v \in V_i\colon\ |E_{defect}(v)| \geq \beta n^3$). As we are not in Case 1, we know that $\forall v \in V\colon\ |E_{bad}(v)| < \beta n^3$.

Case 2 asserts a vertex $v \in V$ such that $|E_{defect}(v)| \geq \beta n^3$ with respect to the partition $\mathcal{P}$ with $V(H) = V_1 \dot\cup V_2 \dot\cup V_3 \dot\cup V_4$. There are exactly 3 types of defective hyperedges incident to vertex v. Therefore, there exists a type τ such that $|E^{\tau}(v)| \geq \beta n^3/3$. Without loss of generality we ssume that $v \in V_1$ and $\tau = (2,1,1,0)$. Recall that this means, that any defective hyperedge of type τ incident to vertex v intersects class V_1 in another vertex distinct from v, and intersects also classes V_2 and V_3, but does not intersect class V_4, as $\tau(4) = 0$. By the minimality of $e(H) - e_H(V_1,V_2,V_3,V_4)$, we know that

$$|E_{\mathrm{cross}}(v)| \geq \beta n^3/3, \tag{5.45}$$

otherwise, moving vertex v to class V_4 would increase the number of crossing hyperedges.

We also note that out of the 20 possible types τ of hyperedges incident to vertex v, we are left to consider only four, namely, the 3 types of defective and one type of crossing hyperedges. The amount of the other 16 types is less than βn^3.

We distinguish between two subsets of the set $\mathcal{C}$ of "allowed" colorings of the set of hyperedges of H. Let $\mathcal{C}_1$ consist of those hyperedge colorings such that there exist two distinct types τ_1 and τ_2, either defective or crossing, with the following property: there exist subsets $E_i(v) \subset E^{\tau_i}(v)$ with $|E_i(v)| \geq \beta n^3/12$ for $i = 1,2$, and both, $E_1(v)$ and $E_2(v)$, are monochromatic in the same color. Moreover, let $\mathcal{C}_2 := \mathcal{C} \setminus \overline{\mathcal{C}_1}$ be the set of remaining colorings.

We first show that $|\mathcal{C}_1| \leq 3^{\mathrm{ex}(n,T_4)-1}$, and then we concentrate on $\mathcal{C}_2$.

Consider a coloring from $\mathcal{C}_1$. By assumption, we always have at least two distinct (defective or crossing) types τ_1 and τ_2 with $|E^{\tau_i}(v)| \geq \beta n^3/3$ for $i = 1,2$. Let us assume that $\tau_1 = \tau$ is the defective type described in the beginning, and let τ_2 be another type. Here we give the arguments only when τ_2 is the crossing type to simplify the presentation. The other cases can be easily treated in a similar way, which will be sketched at the end of this case.

Let $E_i(v) \subset E^{\tau_i}(v)$ with $|E_i(v)| \geq \beta n^3/12$, $i = 1, 2$, be such that all hyperedges in $E_1(v) \dot\cup E_2(v)$ are colored by the same color, say *green*. Each set $E_i(v)$, $i = 1, 2$, can be chosen in at most $\sum_{i=\beta n^3/12}^{n^3} \binom{n^3}{i} \leq 2^{n^3}$ ways. With $|E_1(v)| \geq \beta n^3/12$, and by (5.40) for $0 < \delta \leq (1/96)^4$ there exists a pair $(u, w) \in V_2 \times V_3$ such that v, u, w are contained in at least βn *green* distinct hyperedges intersecting class V_1 in another vertex different from v. We set $X := \{x \colon \{x, u, v, w\} \in E_1(v)\}$. Furthermore, we know that $|E_2(v)| \geq \beta n^3/12$, and hence $|E_2(v) \cap E(V_1, V_2 \setminus \{u\}, V_3 \setminus \{w\}, V_4)| \geq \beta n^3/13$ for n sufficiently large. Thus, there are at least $\beta n^3/13$ green crossing hyperedges incident to vertex v and not containing the vertices u or w. Let f be such a crossing hyperedge and fix one of the at least βn vertices $x \in X$. Then the 4-element set $g := f \setminus \{v\} \dot\cup \{x\}$ together with the hyperedges f and $\{x, v, u, w\}$ forms a subhypergraph T_4 unless g is missing as a hyperedge. For $0 < \delta \leq \beta^2/200$, there are at least

$$(\beta n^3/13)\beta n - \delta n^4 = \beta^2 n^4/13 - \delta n^4 \geq \beta^2 n^4/14 \tag{5.46}$$

many possibilities to choose such a hyperedge $g \in E$. Moreover, g cannot be colored green, thus we only have two remaining colors that can be used. This way, for n sufficiently large, we estimate the cardinality of the set $\mathcal{C}_1$ of colorings for $0 < \delta < \beta^2/40$ as follows:

$$\begin{aligned} |\mathcal{C}_1| &\leq 3 \cdot \binom{4}{2} \cdot 2^{2n^3} \cdot 3^{\mathrm{ex}(n,T_4)+\delta n^4 - \beta^2 n^4/14} \cdot 2^{\beta^2 n^4/14} \\ &\leq 18 \cdot 2^{2n^3} \cdot 3^{\mathrm{ex}(n,T_4)+\delta n^4 - \beta^2 n^4/14} \cdot 2^{\beta^2 n^4/14} \leq 3^{\mathrm{ex}(n,T_4)-1}, \end{aligned} \tag{5.47}$$

taking into account $\binom{4}{2}$ possibilities to choose the types τ_1 and τ_2, and 3 possibilities to choose the color of the hyperedges in the sets $E_1(v)$ and $E_2(v)$, where we used (5.39).

We now consider the colorings in $\mathcal{C}_2$. The most important observation is that, whenever we consider two different (defective or crossing) types τ_1 and τ_2 of hyperedges incident to vertex v, less than $\beta n^3/12$ of the hyperedges from $E^{\tau_i}(v)$ can be colored by the same color for each $i \in [2]$. On the other hand, there are at least two (defective or crossing) types τ_1 and τ_2 for which $|E^{\tau_i}(v)| \geq \beta n^3/3$. Thus, for each of these types τ_i, $i = 1, 2$, there is a color c_i, which occurs at least $\beta n^3/12$ often, where $c_1 \neq c_2$. But then for each other (defective or crossing) type τ_3 or τ_4 each color c_1 and c_2 must occur less than $\beta n^3/12$ often. Moreover, the third color c, $c \neq c_1, c_2$, may occur at least $\beta n^3/12$ in at most one of the types. If this happens for type τ_1 or τ_2, then taking into account the at most βn^3 bad hyperedges incident to vertex v, there are at most

$$\binom{4}{2} \cdot 6 \cdot 2 \cdot 3^{\beta n^3} \cdot 2^{(n/4+3\delta^{1/4}n)^3} \cdot \binom{n^3}{\beta n^3/12}^9 \tag{5.48}$$

colorings of the set of hyperedges of all types incident to vertex v, where we used

$$\sum_{0 \leq i < \beta n^3/12} \binom{\binom{n}{3}}{i} \leq \binom{n^3}{\beta n^3/12}.$$

Moreover, if color c occurs at least $\beta n^3/12$ often in type either τ_3 or τ_4, then there are at most

$$\binom{4}{3} \cdot 3! \cdot 3^{\beta n^3} \cdot \binom{n^3}{\beta n^3/12}^9 \tag{5.49}$$

such colorings of all types of hyperedges incident to vertex v. Note that we first "choose" three types where some colors are present at least $\beta n^3/12$ times and then we assign three colors to these types. Similarly it was argued in (5.48).

Thus, for n sufficiently large by (5.48) and (5.49) and our choice of the parameters $\beta, \delta > 0$ in (5.41), we can estimate by using $2^{11/7} < 3$ the number of ways the set of hyperedges incident to vertex v can be colored by at most

$$73 \cdot 3^{\beta n^3} \cdot 2^{(n/4+3\delta^{1/4}n)^3} \cdot \binom{n^3}{\beta n^3/12}^9 \overset{(5.42),(1.9)}{\leq} 3^{\delta(\mathcal{T}_4^{(4)}(n))-1}. \tag{5.50}$$

Consequently, deleting the vertex v and all hyperedges incident to v we obtain the hypergraph $H' = H - \{v\}$ with

$$c_{3,T_4}(H') \geq \frac{3^{\mathrm{ex}(n,T_4)+m}}{3^{\delta(\mathcal{T}_4^{(4)}(n))-1}} = 3^{\mathrm{ex}(n-1,T_4)+m+1},$$

which yields (5.36) and concludes Case 2 for τ_1 and τ_2 as defined above.

Now assume that both types τ_1 and τ_2 are defective. For convenience, let without loss of generality $\tau_1 = (2,1,1,0)$ and $\tau_2 = (2,0,1,1)$. Similarly, we define the sets $E_i(v) \subset E^{\tau_i}(v)$, $i = 1,2$, of hyperedges of the same color, but now we fix for the type τ_1 a pair $(u,w) \in V_1 \times V_3$ such that v,u,w are contained in at least βn *green* distinct hyperedges intersecting class V_2. Again, let $x \in V_2$ be such a vertex that forms a green hyperedge $\{v,x,u,w\}$, then it is not hard to see that $f \in E_2(v)$ together with $f\setminus\{v\}\cup\{x\}$ and $\{v,x,u,w\}$ form a potential copy of T_4. The rest of the argument remains valid.

Case 3 (H satisfies $\forall i \in \{1,2,3,4\}$ and $\forall v \in V_i$: $|E_{bad}(v)\dot\cup E_{defect}(v)| \leq 2\beta n^3$). Here we are left with the last case, when Cases 1 and 2 do not hold, hence most of the hyperedges incident to *any* vertex v are crossing. By assumption $H \neq T_4^{(4)}(n)$, hence there exists at least one non-crossing hyperedge e with respect to the minimal partition $\mathcal{P}$ with $V(H) = V_1\dot\cup V_2\dot\cup V_3\dot\cup V_4$. Let u,v be two vertices that belong to this hyperedge e and are contained in the same class. Recalling the minimum degree condition (5.37) for H, we infer with (5.40) that

$$|L_{\mathrm{cross}}(u) \cap L_{\mathrm{cross}}(v)| \geq 2\lfloor n/4\rfloor^3 - 4\beta n^3 - (n/4+3\delta^{1/4}n)^3, \tag{5.51}$$

where for a vertex $w \in V$ it is $L_{\mathrm{cross}}(w) := \{e \setminus \{w\} \colon e \in E_{\mathrm{cross}}(w)\}$.

Subtracting from the right hand side of (5.51) the term n^2, which is an upper bound on the number of triples in $L_{\mathrm{cross}}(u)\cap L_{\mathrm{cross}}(v)$ that intersect the hyperedge e in a vertex different from u and v, this way, for n sufficiently large, we have identified at least

$$2\lfloor n/4\rfloor^3 - 4\beta n^3 - (n/4+3\delta^{1/4}n)^3 - n^2$$

copies of T_4, each two distinct of these sharing only the hyperedge e. We have for $0 < \delta \leq (2\beta)^4$ and $\delta \leq (1/12)^4$ and n sufficiently large:

$$2\lfloor n/4 \rfloor^3 - 4\beta n^3 - (n/4 + 3\delta^{1/4} n)^3 - n^2 \geq (n/4)^3 - 8\beta n^3. \tag{5.52}$$

Given the color of the hyperedge e, the two other hyperedges of a fixed copy of T_4 may be colored in at most 8 instead of 9 ways. Therefore, for n sufficiently large, by our choice (5.41) of $\beta, \delta > 0$ (see also the hierarchy (5.42)) we may estimate the number of ways of coloring all hyperedges incident to vertex u or v from above by

$$3 \cdot 3^{4\beta n^3} \cdot 8^{(n/4)^3 - 8\beta n^3} \cdot 3^{2[(n/4+3\delta^{1/4}n)^3 - (n/4)^3 + 8\beta n^3]} \leq 3^{\delta(\mathcal{T}_4^{(4)}(n)) + \delta(\mathcal{T}_4^{(4)}(n-1)) - 1}. \tag{5.53}$$

Again, if we delete the vertices u and v, we obtain the hypergraph $H' = H - \{u, v\}$, and, by a simple averaging argument, it follows that

$$c_{3,T_4}(H') \geq 3^{\mathrm{ex}(n-2, T_4) + m + 1}.$$

This finishes the proof of Claim 5.5 and hence of Theorem 1.13. □

5.2.4 Expanded complete graph and Fan(k)-hypergraph

Here we apply Theorem 1.11 for the hypergraph F being either $H^k_{\ell+1}$ or $F^k_{\ell+1}$ and from the 1-stability for F it follows for a hypergraph H from Theorem 1.11 that one can add or delete up to δn^k hyperedges to obtain a hypergraph which is isomorphic to $\mathcal{T}_\ell^{(k)}(n)$. Due to the structure of $\mathcal{T}_\ell^{(k)}(n)$, this implies that there exists a partition $\mathcal{P}$ of $V(H) = V_1 \dot\cup \ldots \dot\cup V_\ell$ (in a particular a partition that minimizes the number of non-crossing hyperedges in H) such that

$$e_{\mathrm{noncross}}(\mathcal{P}) < \delta n^k. \tag{5.54}$$

Proof of Theorem 1.14. Here we only prove the case $r = 3$, as the arguments for $r = 2$ are very similar. Let $2 \leq k \leq \ell$ and let $F = H^k_{\ell+1}$ or $F = F^k_{\ell+1}$, unless otherwise specified.

Let n_0 be given by Theorem 1.11 (applied with δ, which will be specified later) and let $n_r(F) = n_3(F) \geq n_0$ be sufficiently large.

The proof proceeds by contradiction as follows. Assume that we are given a hypergraph H on $n > n_3$ vertices with $c_{3,F}(H) \geq 3^{\mathrm{ex}(n,F)+m}$ for some $m \geq 0$. We show the following lemma, which is central in our considerations.

Lemma 5.6. *Let $F = F^k_{\ell+1}$ or $F = H^k_{\ell+1}$.*

If $c_{3,F}(H) \geq 3^{\mathrm{ex}(n,F)+m}$ for some $m \geq 0$ and H is not the Turán hypergraph $\mathcal{T}_\ell^{(k)}(n)$, then there exists an induced subhypergraph H' of H on n' vertices with $n' \geq n-2$ and

$$c_{3,F}(H') \geq 3^{\mathrm{ex}(n',F)+m+1}. \tag{5.55}$$

Inductively, for n sufficiently large, we arrive at some subhypergraph H_0 on at most n_0 vertices that admits at least $3^{\text{ex}(n_0,F)+\binom{n_0}{k}+1}$ many F-free 3-colorings of its set of hyperedges, which is impossible and yields a contradiction. Thus, it is left to verify Lemma 5.6, however, this is the major part of the proof. □

Proof of Lemma 5.6. Let $F = F^k_{\ell+1}$ or $F = H^k_{\ell+1}$, $2 \leq k \leq \ell$. Let H be a k-uniform hypergraph on n vertices, $H \neq \mathcal{T}_\ell^{(k)}(n)$ and let $c_{3,F}(H) \geq 3^{\text{ex}(n,F)+m}$ with $m \geq 0$, which implies $e(H) \geq e(\mathcal{T}_\ell^{(k)}(n))$.

Without loss of generality we may assume that the minimum degrees of the hypergraph H and the Turán hypergraph $\mathcal{T}_\ell^{(k)}(n)$ satisfy

$$\delta(H) \geq \delta(\mathcal{T}_\ell^{(k)}(n)) \tag{5.56}$$

Otherwise, let v be a vertex of minimum degree in H and consider the subhypergraph $H' := H - \{v\}$. Since $e(\mathcal{T}(n-1)) = e(\mathcal{T}_\ell^{(k)}(n)) - \delta(\mathcal{T}_\ell^{(k)}(n)) \leq e(\mathcal{T}_\ell^{(k)}(n)) - (\delta(H)+1)$ we infer

$$c_{3,F}(H') \geq \frac{c_{3,F}(H)}{3^{\delta(H)}} = 3^{e(\mathcal{T}_\ell^{(k)}(n))-\delta(H)+m} \geq 3^{e(\mathcal{T}_\ell^{(k)}(n-1))+m+1}, \tag{5.57}$$

that is, by deleting vertex v from H and all hyperedges incident to it, by a simple averaging argument we obtain (5.57), and hence (5.55).

Consider a partition $\mathcal{P}$ with $V(H) = V_1 \dot\cup \cdots \dot\cup V_\ell$, that maximizes the number of crossing hyperedges in H. Let $e_{\text{cross}}(\mathcal{P})$ be this maximum number of crossing hyperedges in H. Therefore, this partition $\mathcal{P}$ minimizes the total number of bad and defective hyperedges. Moreover, since $c_{3,F}(H) \geq 3^{\text{ex}(n,F)}$ by assumption, by Theorem 1.11 with $\ell = \ell(F)$ we know, that for our choice of $\delta > 0$ we have

$$e(H) - e_{\text{cross}}(\mathcal{P}) < \delta \cdot n^k, \tag{5.58}$$

which gives an upper bound on the number of hyperedges in H. Since $e(H) \geq e(\mathcal{T}_\ell^{(k)}(n))$, we have $e_{cross}(\mathcal{P}) > e(\mathcal{T}_\ell^{(k)}(n)) - \delta \cdot n^k$, i.e., less than $\delta \cdot n^k$ crossing hyperedges are missing in H. With $e(H) \geq e(\mathcal{T}_\ell^{(k)}(n))$ and (5.58) it is an easy but tedious calculation to show that for each $i \in [\ell]$ it is

$$n/\ell - (\ell-1) \cdot \delta^{1/k} \cdot n \leq |V_i| \leq n/\ell + \ell^2 \cdot \delta^{1/k} \cdot n. \tag{5.59}$$

In the following our argument splits into three cases depending on the link hypergraph of a vertex. First we assume that there exists a vertex v incident to at least $\beta \cdot n^{k-1}$ *bad* hyperedges with respect to the partition $\mathcal{P}$ (Case 1). If this is not the case, then we assume that there exists a vertex v, which is incident to at least $\beta \cdot n^{k-1}$ *defective* hyperedges with respect to the partition $\mathcal{P}$ (Case 2). Finally, if neither Case 1 nor Case 2 holds, we deal with Case 3, where every vertex is adjacent to at most $2 \cdot \beta \cdot n^{k-1}$ many *defective* or *bad* hyperedges with respect to the partition $\mathcal{P}$. Thus, in Case 3 by the assumption (5.56) on the high minimum degree, with $0 < \beta \ll 1$ we know that every vertex is adjacent mostly to *crossing* hyperedges with respect to the partition $\mathcal{P}$.

We will omit the explicit setting of the small parameters δ, β and also ε as well (which appears at a later stage of the proof). It is sufficient to keep in mind that

$$0 < \delta \ll \beta \ll \varepsilon \ll 1. \tag{5.60}$$

Obviously, for the partition $\mathcal{P}$ and any vertex v there are exactly

- $c_\ell := \binom{\ell-1}{k-1}$ types of crossing hyperedges incident to v, and
- $d_\ell := \binom{\ell-1}{k-2}$ types of defective hyperedges incident to v, and
- $b_\ell := \binom{k+\ell-2}{k-1} - \binom{\ell}{k-1}$ types of bad hyperedges incident to v.

The further organization of the proof is, that every case is presented in its own subsection.

Case 1: H satisfies $\exists i \in [\ell]$ and $\exists v \in V_i\colon\ |E_{\text{bad}}(v)| \geq \beta \cdot n^{k-1}$

Recall, that $e(H) \geq \text{ex}(n, F)$ and that by assumption the hypergraph H is not isomorphic to the Turán hypergraph $\mathcal{T}_\ell^{(k)}(n)$, thus H contains at least one subhypergraph F. Indeed, by the assumption $c_{3,F}(H) \geq 3^{\text{ex}(n,F)+m}$ for some $m \geq 0$, it turns out that Case 1 never holds for appropriately chosen small $\beta > 0$.

Assume without loss of generality that $i = 1$, and let $v \in V_1$ be a vertex such that $|E_{\text{bad}}(v)| \geq \beta \cdot n^{k-1}$. There are at most b_ℓ types of *bad* hyperedges incident to vertex v. Thus, for at least one type τ we know that $|E^\tau(v)| \geq \beta \cdot n^{k-1}/b_\ell$. By an averaging argument, there exist $(k-3)$ distinct vertices $w_1, \ldots, w_{k-3}$, all distinct from vertex v, such that the common link graph

$$L(v, w_1, \ldots, w_{k-3}) = \{e \setminus \{v, w_1, \ldots, w_{k-3}\}\colon\ e \in E^\tau(v) \text{ and } w_1, \ldots, w_{k-3} \in e\}$$

contains at least $\beta \cdot n^2/b_\ell$ edges, which all are contained in some class V_j, $j \in [\ell]$, that is, together with any edge from $L(v, w_1, \ldots, w_{k-3})$, the vertices v and $w_1, \ldots, w_{k-3}$ form a hyperedge of type τ in H. Then, greedily we can find a matching $M \subseteq L(v, w_1, \ldots, w_{k-3})$, hence $M \subseteq [V_j]^2$, of size $m \geq \beta \cdot n/(2 \cdot b_\ell)$. Let

$$M := \{\{a_1, b_1\}, \ldots, \{a_m, b_m\}\}$$

and $e_s := \{a_s, b_s\}$, $s \in [m]$, where without loss of generality $m \leq n/(5 \cdot \ell)$, otherwise we delete some edges from M.

First we deal with the case when $F = H_{\ell+1}^k$. We know by (5.59) that, for $0 < \delta \leq (1/(5 \cdot \ell^3))^k$ and for n sufficiently large, every class V_i, $i \in [\ell]$, has size at least $4 \cdot n/(5 \cdot \ell)$, thus we may select from every class V_i, $i \neq j$, two disjoint subsets A_i and B_i each of size $n/(3 \cdot \ell)$, where both A_i and B_i are disjoint from the set $\{v, w_1, \ldots, w_{k-3}\}$. Moreover, we define for the class V_j the sets $A_j := \{a_1, \ldots, a_m\}$ and $A_j^* := \{b_1, \ldots, b_m\}$ and a subset $B_j \subset V_j$ of size $n/(3 \cdot \ell)$, which is disjoint from $A_j \cup A_j^* \cup \{v, w_1, \ldots, w_{k-3}\}$.

We want to find $\Theta(n^k)$ copies of $H_{\ell+1}^k$ (these need not be subhypergraphs $H_{\ell+1}^k$ in H, as some hyperedges might be missing), such that, on average, only $\Theta(n^{k-1})$ of the copies share some hyperedge (which contains some matching edge from M), and moreover these copies are "almost" hyperedge-disjoint from other copies. More precisely, we show:

Lemma 5.7. *There exists a family $\mathcal{F}$ of subhypergraphs $H^k_{\ell+1}$ in H with the following properties:*

- $\mathcal{F} = \mathcal{F}_1 \dot\cup \cdots \dot\cup \mathcal{F}_m$ *with* $|M| = m \geq \beta \cdot n/(2 \cdot b_\ell)$, *and*
- $|\mathcal{F}| \geq \frac{\beta}{3^{k+\ell-3}\ell^{\ell+3k-5} \cdot 4 \cdot b_\ell} \cdot n^k$, *and*
- *for all* $F_1 \in \mathcal{F}_s$ *and* $F_2 \in \mathcal{F}_t$ *with* $s \neq t$ *it is* $E(F_1) \cap E(F_2) = \varnothing$, *i.e., any two subhypergraphs* $H^k_{\ell+1}$ *from different subfamilies do not have any hyperedges in common, and*
- *for all* $F_1, F_2 \in \mathcal{F}_s$, $s \in [m]$, *it is* $E(F_1) \cap E(F_2) = e = \{v, w_1, \ldots, w_{k-3}\} \dot\cup e_s$ *with* $e_s \in M$ *and* $e \in E$, *and we call the hyperedge* e *the* common *hyperedge of the subfamily* $\mathcal{F}_s$.

Proof. We use in our arguments the following simple claim:

Claim 5.8. *Let G be the complete, r-partite, r-uniform hypergraph with classes of sizes $c_i \cdot N$ for constants $0 < c_i \leq 1$, $i \in [r]$.*

Then, there exists a linear subhypergraph $\mathcal{G}$ of G with at least $(N^2/r^2) \cdot \prod_{i=1}^{r} c_i$ hyperedges.

Proof. Given the complete, r-partite, r-uniform hypergraph as specified in the assumption, we start by picking any hyperedge f from G and delete all hyperedges from G that intersect f in at least 2 vertices. Then, we repeat this procedure with the resulting subhypergraph until the remaining hyperedges pairwise meet in at most one vertex. In each step we delete at most $\binom{r}{2} \cdot N^{r-2}$ hyperedges. This way, we clearly find a linear subhypergraph $\mathcal{G}$ with at least $(N^2/r^2) \cdot \prod_{i=1}^{r} c_i$ hyperedges. □

For $r = \ell$, we apply Claim 5.8 to the complete, ℓ-partite, ℓ-uniform hypergraph with vertex classes $A_1, \ldots, A_\ell$, where $|A_i| = n/(3 \cdot \ell)$ for each $i \neq j$ and $|A_j| = m \geq \beta \cdot n/(2 \cdot b_\ell)$, and we obtain a linear family $\mathcal{G}$ on $A_1 \dot\cup \cdots \dot\cup A_\ell$ with

$$|\mathcal{G}| \geq \frac{\beta}{2 \cdot \ell^2 \cdot b_\ell} \cdot \left(\frac{1}{3 \cdot \ell}\right)^{\ell-1} \cdot n^2. \tag{5.61}$$

Let $B = B_1 \dot\cup \cdots \dot\cup B_\ell$, i.e., $|B| \geq n/3$ since $|B_i| \geq n/(3 \cdot \ell)$, $i \in [\ell]$, and note that by the choice of the sets A_i, B_i, $i \in [\ell]$, we have $B \cap (A_1 \cup \cdots \cup A_\ell) = \varnothing$. Partition the set B into $(\binom{\ell+1}{2} - 1)$ mutually disjoint subsets $B_{(x,y)}$, $1 \leq x < y \leq \ell+1$ but $(x, y) \neq (j, j+1)$, each of size

$$|B_{(x,y)}| \geq \frac{n}{3 \cdot \ell^2} \quad \text{such that} \quad |B_{(x,y)} \cap B_i| \geq \frac{n}{3 \cdot \ell^3} \text{ for all } i \in [\ell],$$

and set

$$T := \left(\frac{n}{3 \cdot \ell^3}\right)^{k-2}. \tag{5.62}$$

For each pair (x,y), $1 \leq x < y \leq \ell+1$ but $(x,y) \neq (j,j+1)$, choose T pairwise distinct $(k-2)$-element subsets from the set $B_{(x,y)}$, crossing with respect to the partition $\mathcal{P}$, and enumerate these as $u_{(x,y)}(1), \ldots, u_{(x,y)}(T)$.

With any hyperedge $e \in \mathcal{G}$ we associate an ℓ-tuple $\hat{e}$, such that for $\hat{e} = (v_1, \ldots, v_\ell)$, we have $e = \{v_1, \ldots, v_\ell\} \in \mathcal{G}$, where $v_i \in A_i$ for $i \in [\ell]$ (and therefore $v_j = a_s$ for some $s \in [m]$). We enlarge $\hat{e}$ by the vertex b_s to $\hat{e}^* = (v_1, \ldots, v_{j-1}, a_s, b_s, v_{j+1}, \ldots, v_\ell)$, where $\{a_s, b_s\}$ is an edge from the matching M. Let $(\hat{e}^*)_i$ denote the entry of $\hat{e}^*$ in coordinate i, $i \in [\ell+1]$. Fix some integer $p \in [T]$. For every pair (x,y) of integers, $1 \leq x < y \leq \ell+1$ but $(x,y) \neq (j,j+1)$, we enlarge the 2-element set $\{(\hat{e}^*)_x, (\hat{e}^*)_y\}$ by the $(k-2)$-element set $u_{(x,y)}(p)$ to a k-element set. Moreover, we extend the 2-element set $\{a_s, b_s\}$ by the $(k-2)$-element set $\{v, w_1, \ldots, w_{k-3}\}$, which is a hyperedge in H. These $\binom{\ell+1}{2}$ many $(k-2)$-element sets $u_{(x,y)}(p)$ and $\{v, w_1, \ldots, w_{k-3}\}$ are pairwise disjoint by construction, and we obtain a copy $H(\hat{e}, p)$ of $H^k_{\ell+1}$ with core $\hat{e}^*$ in the complete ℓ-partite k-uniform hypergraph $K[V_1, \ldots, V_\ell]$ (on the same vertex set as H).

We construct this way such copies $H(\hat{e}, p)$ of $H^k_{\ell+1}$ for every $e \in \mathcal{G}$ and every $p \in [T]$. For $s \in [m]$, define the families $\mathcal{F}_s := \{H(\hat{e}, p) : e \in \mathcal{G}, p \in [T], (\hat{e})_j = a_s\}$.

We claim that distinct copies $H(\hat{e}, p)$ and $H(\hat{e}', p')$ of $H^k_{\ell+1}$ from the same subfamily $\mathcal{F}_s$ intersect in the k-element set $\{a_s, b_s, v, w_1, \ldots, w_{k-3}\} \in E$ only, while copies $H(\hat{e}, p)$ and $H(\hat{e}', p')$ of $H^k_{\ell+1}$ from distinct subfamilies $\mathcal{F}_s$ and $\mathcal{F}_t$, $s \neq t$, respectively, do not have any k-element set in common.

Namely, if $e, e' \in \mathcal{G}$, where $(\hat{e})_j \neq (\hat{e}')_j$, then by construction $|e \cap e'| \leq 1$, and thus for any $p, p' \in [1, T]$ the copies $H(\hat{e}, p)$ and $H(\hat{e}', p')$ of $H^k_{\ell+1}$ do not have any k-element set in common.

Now let $e, e' \in \mathcal{G}$ with $(\hat{e})_j = (\hat{e}')_j = a_s$. If $e = e'$, and $p \neq p'$, then the copies $H(\hat{e}, p)$ and $H(\hat{e}, p')$ of $H^k_{\ell+1}$ intersect in the k-element set $\{a_s, b_s, v, w_1, \ldots, w_{k-3}\}$ only, as $u_{(x,y)}(p) \neq u_{(x,y)}(p')$ for all (x,y), $1 \leq x < y \leq \ell+1$ and $(x,y) \neq (j,j+1)$, and as the sets $B_{(x,y)}$ are pairwise disjoint. If $e \neq e'$ and $(\hat{e})_j = (\hat{e}')_j = a_s$, then for any $p, p' \in [T]$, with $|e \cap e'| = 1$ again we infer that the copies $H(\hat{e}, p)$ and $H(\hat{e}, p')$ of $H^k_{\ell+1}$ only intersect in the k-element set $\{a_s, b_s, v, w_1, \ldots, w_{k-3}\}$, which is a hyperedge in H.

Thus, using (5.61) and (5.62) we have found at least

$$T \cdot |\mathcal{G}| \geq \left(\frac{n}{3 \cdot \ell^3}\right)^{k-2} \cdot \left(\frac{1}{3 \cdot \ell}\right)^{\ell-1} \cdot \frac{\beta}{2 \cdot \ell^2 \cdot b_\ell} \cdot n^2 = \frac{\beta}{3^{k+\ell-3} \cdot \ell^{\ell+3k-5} \cdot 2 \cdot b_\ell} \cdot n^k$$

copies of $H^k_{\ell+1}$ in $K[V_1, \ldots, V_\ell]$. However, not all of these copies of $H^k_{\ell+1}$ might be present in H as subhypergraphs, as by (5.58) at most $\delta \cdot n^k$ crossing hyperedges are missing in H. But as all common hyperedges $\{a_s, b_s, v, w_1, \ldots, w_{k-3}\}$, $s \in [m]$, are present in H, we obtain for $0 < \delta \ll \beta$ at least

$$\frac{\beta}{3^{k+\ell-3} \cdot \ell^{\ell+3k-5} \cdot 2 \cdot b_\ell} \cdot n^k - \delta \cdot n^k \overset{(5.60)}{\geq} \frac{\beta}{3^{k+\ell-3} \cdot \ell^{\ell+3k-5} \cdot 4 \cdot b_\ell} \cdot n^k$$

subhypergraphs $H^k_{\ell+1}$ in H with the desired properties, as claimed in Lemma 5.7. $\square$

Next we consider the case $F = F^k_{\ell+1}$.

Lemma 5.9. *There exists a family $\mathcal{F}$ of subhypergraphs $F^k_{\ell+1}$ in the hypergraph H with the following properties:*

- $\mathcal{F} = \mathcal{F}_1 \dot\cup \cdots \dot\cup \mathcal{F}_m$ *with* $|M| = m = \beta \cdot n/(2 \cdot b_\ell)$*, and*
- $|\mathcal{F}| \geq \frac{1}{53 \cdot \binom{\ell}{k} \cdot (2 \cdot b_\ell)^k} \cdot \beta^k \cdot n^k$*, and*
- *for all* $F_1 \in \mathcal{F}_s$ *and* $F_2 \in \mathcal{F}_t$ *with* $s \neq t$ *it is* $E(F_1) \cap E(F_2) = \varnothing$*, i.e., any two subhypergraphs* $F^k_{\ell+1}$ *from different subfamilies do not have any hyperedges in common, and*
- *for all* $F_1, F_2 \in \mathcal{F}_s$*,* $s \in [m]$*, it is* $E(F_1) \cap E(F_2) = e = \{v, w_1, \ldots, w_{k-3}\} \dot\cup e_s$ *with* $e_s \in M$ *and* $e \in E$*, and we call the hyperedge* e *the* common *hyperedge of the subfamily* $\mathcal{F}_s$*.*

Proof. For the proof we use the following claim

Claim 5.10. *Let $r \geq k \geq 3$ be integers. Let $0 < c < 1/(12 \cdot \binom{r}{k})$ be a constant. Let G be the complete, r-partite, r-uniform hypergraph with classes each of size N.*

Then, there exists a subhypergraph $\mathcal{G}$ of G with at least $c \cdot N^k/4$ hyperedges such that

- *distinct hyperedges* $e, e' \in E(\mathcal{G})$ *do not have any k-element subset in common, and*
- *for each two-element set* $\{v, w\}$ *of vertices in* $V(\mathcal{G})$ *there are at most* $2 \cdot c \cdot N^{k-2}$ *hyperedges* $e \in E(\mathcal{G})$*, which contain* $\{v, w\}$*.*

Proof. We show the existence of the subhypergraph $\mathcal{G}$ by a probabilistic argument.

With probability $p = c/N^{r-k}$ for some constant $c > 0$ we pick uniformly at random and independently of each other hyperedges from G. Let S be the random variable counting the number of chosen hyperedges. Then, the expected number $E[S]$ satisfies

$$\mathbb{E}[S] = p \cdot N^r = c \cdot N^k.$$

The random variable S is binomially distributed hence by Chernoff's inequality, Theorem 2.19, i.e., $\mathbb{P}(\mathbb{E}[S] - S > \alpha \cdot \mathbb{E}[S]) \leq e^{-\alpha^2 \mathbb{E}[S]/2}$ for $0 < \alpha < 1$, we infer for N large enough

$$\mathbb{P}(S < c \cdot N^k/2) < e^{-cN^k/8} < \frac{1}{3}. \tag{5.63}$$

Let P count the number of pairs of chosen r-element subsets which have a k-element set in common. Then we infer for the expectation $\mathbb{E}[P]$

$$\mathbb{E}[P] \leq \binom{r}{k} \cdot N^k \cdot (N^{r-k})^2 \cdot p^2 = c^2 \cdot \binom{r}{k} \cdot N^k,$$

and by Markov's inequality we have

$$\mathbb{P}(P > 3 \cdot c^2 \cdot \binom{r}{k} \cdot N^k) < \frac{1}{3}. \tag{5.64}$$

Now, for a fixed pair $\{v, w\}$ of vertices from different classes, let $R_{v,w}$ be the random variable counting the number of r-element sets, which contain both vertices v and w, hence

$$\mathbb{E}[R_{v,w}] = p \cdot N^{r-2} = c \cdot N^{k-2}.$$

The random variable $R_{v,w}$ is binomially distributed, hence by Chernoff's inequality, Theorem 2.19, we infer

$$\mathbb{P}(R_{v,w} > 2 \cdot E[R_{v,w}]) < e^{-(3c/8)N^{k-2}}.$$

From this we conclude for the probability that there exists a pair $\{v, w\}$ of vertices from different classes which are contained in more than $2 \cdot E[R_{v,w}]$ random r-element sets, the upper bound

$$\binom{r}{2} \cdot N^2 \cdot e^{-cN^{k-2}} < \frac{1}{3} \tag{5.65}$$

for n large enough.

By (5.63)–(5.65) we infer that there exists a family $\mathcal{G}'$ of r-element subsets of size at least $c \cdot N^k/2$, where the number of distinct r-element sets in $\mathcal{G}'$, which have a k-element set in common, is at most $3 \cdot c^2 \cdot \binom{r}{k} \cdot N^k$ and where every pair $\{v, w\}$ of distinct vertices is contained in at most $2 \cdot c \cdot N^{k-2}$ r-element sets of $\mathcal{G}'$.

For $c < 1/(12 \cdot \binom{r}{k})$ we delete from each pair of distinct r-element sets, which have a k-element set in common, one of the r-element sets, and we obtain a subfamily $\mathcal{G} \subseteq \mathcal{G}'$ of size at least $c \cdot N^k/4$ r-sets, which pairwise do not have a k-element set in common and where every pair $\{v, w\}$ of distinct vertices is contained in at most $2 \cdot c \cdot N^{k-2}$ r-element sets of $\mathcal{G}$. $\square$

We know by (5.59) that, for $0 < \delta \leq (1/(5 \cdot \ell^3))^k$ and for n sufficiently large, every class V_i, $i \in [\ell]$, has size at least $4 \cdot n/(5 \cdot \ell)$, thus we may select from every class V_i, $i \neq j$, two disjoint subsets A_i, and B_i, where each set A_i has size m, and each set B_i has size $n/(3 \cdot \ell)$, and both A_i and B_i are disjoint from the set $\{v, w_1, \ldots, w_{k-3}\}$. Moreover, we select from the class V_j the sets $A_j = \{a_1, \ldots, a_m\}$ and $A_j^* = \{b_1, \ldots, b_m\}$ and a subset $B_j \subset V_j$ of size $n/(3 \cdot \ell)$, which is disjoint from $A_j \cup A_j^* \cup \{v, w_1, \ldots, w_{k-3}\}$.

For $r = \ell$, we apply Claim 5.10 to the complete, ℓ-partite, ℓ-uniform hypergraph with vertex classes $A_1, \ldots, A_\ell$, where $|A_i| = m := \beta \cdot n/(2 \cdot b_\ell)$, with $c = 1/(13 \cdot \binom{\ell}{k})$ and we obtain a family $\mathcal{G}$ on $A_1 \dot\cup \cdots \dot\cup A_\ell$ of crossing ℓ-element sets with

$$|\mathcal{G}| = \frac{1}{52 \cdot \binom{\ell}{k}} \cdot m^k,$$

where pairwise the hyperedges in $\mathcal{G}$ do not have any k-element subset in common, and where every pair v, w of distinct vertices in $V(\mathcal{G})$ is contained in at most $(2/(13\binom{\ell}{k})) \cdot m^{k-2}$ many hyperedges from $\mathcal{G}$.

For each hyperedge g in $\mathcal{G}$, we construct copies of the hypergraph $F_{\ell+1}^k$ with core containing the set g. Enumerate the ℓ-element sets (hyperedges) in $\mathcal{G}$ by $g_1, \ldots, g_{(1/(52\binom{\ell}{k}))m^k}$.

For each ℓ-element set g_i, $i \in [(1/(52 \cdot \binom{\ell}{k})) \cdot m^k]$ choose a k-element subset K_i with $|K_i \cap A_j| = 1$. This k-element set is the core-hyperedge of g_i.

Let $B = B_1 \dot\cup \cdots \dot\cup B_\ell$, i.e., $|B| \geq n/3$ since $|B_i| \geq n/(3 \cdot \ell)$, $i \in [\ell]$, and note that $B \cap (A_1 \cup \cdots \cup A_\ell) = \varnothing$. Partition the set B into $(\binom{\ell+1}{2} - 1)$ mutually disjoint subsets $B_{(x,y)}$, $1 \leq x < y \leq \ell + 1$ but $(x, y) \neq (j, j + 1)$, each of size

$$|B_{(x,y)}| \geq \frac{n}{3 \cdot \ell^2} \quad \text{such that} \quad |B_{(x,y)} \cap B_i| \geq \frac{n}{3 \cdot \ell^3} \text{ for all } i \in [\ell],$$

which leads to at least

$$\left(\frac{n}{3 \cdot \ell^3}\right)^{k-2} \gg \left(\frac{2}{13\binom{\ell}{k}}\right) m^{k-2}$$

distinct crossing $(k-2)$-element sets.

For each pair (x, y), $1 \leq x < y \leq \ell+1$ but $(x, y) \neq (j, j+1)$, choose $(2/(13\binom{\ell}{k})) \cdot m^{k-2}$ pairwise distinct $(k-2)$-element subsets from the set $B_{(x,y)}$, and enumerate these as $u_{(x,y)}(1), \ldots, u_{(x,y)}((2/(13\binom{\ell}{k}))m^{k-2})$.

For each pair v, w of distinct vertices enumerate all ℓ-subsets in $\mathcal{G}$, which contain both vertices v and w by $L_{(v,w)}(1), \ldots, L_{(v,w)}(n(v, w))$, where $n(v, w) \leq 2/(13 \cdot \binom{\ell}{k}) \cdot m^{k-2}$.

With any hyperedge $g_p \in \mathcal{G}$ we associate an ℓ-tuple $\hat{g}_p$, such that for $\hat{g}_p = (v_1, \ldots, v_\ell)$, we have $g_p = \{v_1, \ldots, v_\ell\} \in \mathcal{G}$, where $v_i \in A_i$ for $i \in [\ell]$ (and therefore $v_j = a_s$ for some $s \in [m]$). We enlarge $\hat{g}_p$ by the vertex b_s to $\hat{g}_p^* = (v_1, \ldots, v_{j-1}, a_s, b_s, v_{j+1}, \ldots, v_\ell)$, where $\{a_s, b_s\}$ is an edge from the matching M. Let $(\hat{g}_p^*)_i$ denote the entry of $\hat{g}_p^*$ in coordinate i, $i \in [\ell + 1]$.

For each pair v, w from $\hat{g}_p^*$ of distinct vertices with $v \in A_x$ and $w \in A_y$, $x \neq y$ and $(x, y) \neq (j, j + 1)$, hence $\{v, w\} \neq \{a_s, b_s\}$, we extend the 2-element set $\{v, w\}$ by the $(k-2)$-element set $u_{(x,y)}(i)$, if $g_p = L_{(v,w)}(i)$ for some i and not both vertices v and w are contained in the core hyperedge K_p of g_p. Moreover, we extend the 2-element set $\{a_s, b_s\}$ by the $(k-2)$-element set $\{v, w_1, \ldots, w_{k-3}\}$, which is a hyperedge in H. These $1 + \binom{\ell+1}{2} - \binom{k}{2}$ many $(k-2)$-element sets $u_{(x,y)}(i)$ and $\{v, w_1, \ldots, w_{k-3}\}$ are pairwise disjoint by construction, and we obtain a copy $F(\hat{g}_p)$ of $F^k_{\ell+1}$ with core K_p.

We construct such copies $F(\hat{g}_p)$ of $F^k_{\ell+1}$ for every $g_p \in \mathcal{G}$. For $s \in [m]$, define the families $\mathcal{F}_s := \{F(\hat{g}_p) : g_p \in \mathcal{G}, (\hat{g}_p)_j = a_s\}$.

By construction, distinct copies $F(\hat{g}_p)$ and $F(\hat{g}'_p)$ of $F^k_{\ell+1}$ from the same subfamily $\mathcal{F}_s$ intersect in the k-element set $\{a_s, b_s, v, w_1, \ldots, w_{k-3}\} \in E(H)$ only, while copies $F(\hat{g}_p)$ and $F(\hat{g}'_p)$ of $F^k_{\ell+1}$ from distinct subfamilies $\mathcal{F}_s$ and $\mathcal{F}_t$, $s \neq t$, respectively, do not have any k-element set in common.

Thus, we have found at least

$$|\mathcal{G}| \geq \frac{1}{52 \cdot \binom{\ell}{k}} \cdot m^k \geq \frac{1}{52 \cdot \binom{\ell}{k}} \cdot \left(\frac{\beta \cdot n}{2 \cdot b_\ell}\right)^k = \frac{\beta^k \cdot n^k}{52 \cdot \binom{\ell}{k} \cdot (2 \cdot b_\ell)^k} \tag{5.66}$$

copies of $F^k_{\ell+1}$ in the complete ℓ-partite k-uniform hypergraph $K[V_1, \ldots, V_\ell]$ (on the same vertex set as H). Not all of these copies of $F^k_{\ell+1}$ might be present in H as subhypergraphs, as by (5.58) at most $\delta \cdot n^k$ crossing hyperedges are missing in H. But as all common hyperedges $\{a_s, b_s, v, w_1, \ldots, w_{k-3}\}$, $s \in [m]$, are present in H, we obtain for $0 < \delta \ll \beta$ at least

$$\frac{1}{52 \cdot \binom{\ell}{k} \cdot (2 \cdot b_\ell)^k} \cdot \beta^k \cdot n^k - \delta \cdot n^k \geq \frac{1}{53 \cdot \binom{\ell}{k} \cdot (2 \cdot b_\ell)^k} \cdot \beta^k \cdot n^k$$

subhypergraphs $F^k_{\ell+1}$ in H with the desired properties, as claimed in Lemma 5.9. $\square$

Now for $F = F^k_{\ell+1}$ or $F = H^k_{\ell+1}$ we show how the existence of the family $\mathcal{F}$, as guaranteed by Lemmas 5.7 and 5.9, implies that Case 1 never holds. Let $\mathcal{F}' \subseteq \mathcal{F}$ with $\mathcal{F}'_s \subseteq \mathcal{F}_s$, $s \in [m]$, be a subfamily of size the minimum guaranteed by Lemmas 5.7 and 5.9, i.e., for $0 < \beta \ll 1$

$$|\mathcal{F}'| \geq \frac{1}{53 \cdot \binom{\ell}{k} \cdot (2 \cdot b_\ell)^k} \cdot \beta^k \cdot n^k.$$

In what follows we estimate the number of F-free 3-colorings of the set of hyperedges of H. Once we fix the color of the common hyperedge in a subfamily $\mathcal{F}'_s$, $s \in [m]$, we may color the set of remaining hyperedges of any single subhypergraph F in $\mathcal{F}'_s$ in at most $(3^{e(F)-1} - 1)$ instead of at most $3^{e(F)-1}$ ways, as otherwise we obtain a monochromatic subhypergraph F. Applying the same considerations to all common hyperedges in every subfamily $\mathcal{F}'_s$, $s \in [m]$, and the corresponding subhypergraphs F, we obtain the following possibilities for coloring the set of hyperedges of H:

- every common hyperedge in a subfamily $\mathcal{F}'_s$, $s \in [m]$, may be colored in at most 3 ways, and
- by Lemma 5.9 there exist at least $\frac{1}{53 \cdot \binom{\ell}{k} \cdot (2 \cdot b_\ell)^k} \cdot \beta^k \cdot n^k$ pairwise distinct subhypergraphs F in H, and hence at least

  $$(e(F) - 1) \cdot \frac{1}{53 \cdot \binom{\ell}{k} \cdot (2 \cdot b_\ell)^k} \cdot \beta^k \cdot n^k$$

  pairwise distinct hyperedges in H distinct from the common hyperedges, such that any of the $(e(F) - 1)$ hyperedges of a single subhypergraph F may be colored in at most $(3^{e(F)-1} - 1)$ instead of at most $3^{e(F)-1}$ ways, and
- finally, the set of the remaining hyperedges may be colored arbitrarily by at most 3 colors.

Thus, for $0 < \delta \ll \beta$ and n sufficiently large, we bound from above the number of hyperedge 3-colorings of H by

$$3^{\mathrm{ex}(n,F)+\delta n^k - (e(F)-1)\cdot \frac{1}{53\cdot\binom{\ell}{k}\cdot(2\cdot b_\ell)^k}\cdot\beta^k\cdot n^k} \cdot (3^{e(F)-1} - 1)^{\frac{1}{53\cdot\binom{\ell}{k}\cdot(2\cdot b_\ell)^k}\cdot\beta^k\cdot n^k}$$
$$= 3^{\mathrm{ex}(n,F_{\ell+1}k)+\delta n^k} \cdot \left(\frac{3^{e(F)-1} - 1}{3^{e(F)-1}}\right)^{\frac{1}{53\cdot\binom{\ell}{k}\cdot(2\cdot b_\ell)^k}\cdot\beta^k\cdot n^k} \ll 3^{\mathrm{ex}(n,F^k_{\ell+1})},$$

which contradicts the assumption $c_{3,F}(H) \geq 3^{\mathrm{ex}(n,F)}$. Therefore, for both cases $F = F^k_{\ell+1}$ or $F = H^k_{\ell+1}$, we have shown that Case 1 never holds.

Case 2: H satisfies $\exists i \in [\ell]$ and $\exists v \in V_i\colon\ |E_{\text{defect}}(v)| \geq \beta \cdot n^{k-1}$

In the following we denote $F_{\ell+1} = H^k_{\ell+1}$ or $F_{\ell+1} = F^k_{\ell+1}$.

We introduce some further notation. For a vertex $v \in V_i$, let $e \in E(H)$ (with $v \in e$) be a hyperedge of some type τ with respect to the partition $\mathcal{P}$. If e intersects the class V_i in a further vertex distinct from v, we say that the vertex v *covers* class V_i *via type* τ and hyperedge e, moreover, we say that vertex v covers class V_i via type τ *by* $|E^\tau(v)|$ hyperedges.

As we are not in Case 1, we know that for each $j \in [\ell]$ and for each vertex $v \in V_j$ we have $|E_{\text{bad}}(v)| \leq \beta \cdot n^{k-1}$. Case 2 asserts the existence of a vertex $v \in V$ such that $|E_{\text{defect}}(v)| \geq \beta \cdot n^{k-1}$ with respect to the maximal partition $\mathcal{P}$. Recall that in the hypergraph H there are at most $c_\ell = \binom{\ell-1}{k-1}$ types of crossing hyperedges and at most $d_\ell = \binom{\ell-1}{k-2}$ types of defective hyperedges incident to the vertex v.

Lemma 5.11. *Let $\ell \geq 2$ be a fixed integer. Let $\mathcal{P}$ be a partition with $V(H) = V_1 \dot\cup \cdots \dot\cup V_\ell$, that maximizes the number of crossing hyperedges in the hypergraph H.*

If there exists a vertex v such that $|E_{\text{defect}}(v)| \geq \beta \cdot n^{k-1}$ with respect to the partition $\mathcal{P}$, then v covers each class V_i, $i \in [\ell]$, via some crossing type by at least

$$\beta \cdot n^{k-1}/(d_\ell \cdot c_\ell) \tag{5.67}$$

hyperedges.

Proof. Since $|E_{\text{defect}}(v)| \geq \beta \cdot n^{k-1}$, there exists a defective type τ such that $|E^\tau(v)| \geq \beta \cdot n^{k-1}/d_\ell$. Assume without loss of generality that $v \in V_1$ and all hyperedges of type τ incident to vertex v intersect class V_1 in one further vertex, and each class $V_2, \ldots, V_{k-1}$ in exactly one vertex. Thus, the remaining classes $V_k, \ldots, V_\ell$ are disjoint from hyperedges of type τ incident to vertex v, and v covers via type τ each class $V_1, \ldots, V_{k-1}$ by $|E^\tau(v)| \geq \beta \cdot n^{k-1}/d_\ell$ hyperedges. By assumption, vertex v covers class V_1 by at least $\beta \cdot n^{k-1}$ defective hyperedges. If we would move vertex v to some class V_j with $j \geq k$, then, by the maximality of the partition $\mathcal{P}$, we would not increase the number of crossing hyperedges. Therefore, we conclude that $|E_{\text{cross}}(v)| \geq |E^\tau(v)|$, hence

$$|E_{\text{cross}}(v)| \geq \beta \cdot n^{k-1}/d_\ell. \tag{5.68}$$

Then, there must be a crossing type τ' of hyperedges incident to vertex v, and intersecting class V_j in one vertex, such that $|E^{\tau'}(v)| \geq \beta \cdot n^{k-1}/(c_\ell \cdot d_\ell)$. In fact, we even have $|E^{\tau'}(v)| \geq \beta \cdot n^{k-1}/(d_\ell \cdot \binom{\ell-2}{k-2})$, as moving vertex v to the class V_j would destroy only those types of crossing hyperedges, which are incident to v and intersect class V_j in one vertex. Thus, we have shown that vertex v covers every class V_i, $i \in [\ell]$, by at least $\beta \cdot n^{k-1}/(d_\ell \cdot c_\ell)$ hyperedges, and (5.67) follows. □

Next we partition the set $\mathcal{C}$ of allowed 3-colorings of the set of hyperedges of H into two sets $\mathcal{C}_1$ and $\mathcal{C}_2$, i.e., $\mathcal{C} = \mathcal{C}_1 \dot\cup \mathcal{C}_2$. Having shown that the size of $\mathcal{C}_1$ is small, we then concentrate on $\mathcal{C}_2$ and perform the inductive step.

Let $\mathcal{C}_1$ consist of the set of all 3-colorings of the set of hyperedges of H such that vertex v covers every class $V_i, i \in [\ell]$, via some defective (for $i = 1$) or crossing (for $i \geq 2$) type by at least $\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)$ hyperedges, where all these hyperedges are colored the same, i.e., they are all either blue, green, or red.

Fix some coloring c from $\mathcal{C}_1$. Let $E_i(v)$ be the set of all hyperedges incident to vertex v of some type, defective or crossing, such that v covers class V_i by $E_i(v)$, $i \in [\ell]$, with

$$|E_i(v)| \geq \beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell),$$

and all hyperedges in $\cup_{i=1}^{\ell} E_i(v)$ are colored the same, say, in green. Let

$$H_v = (V, \cup_{i=1}^{\ell} E_i(v))$$

denote the subhypergraph of H, containing all these green hyperedges incident to vertex v.

In the following we show the upper bound $|\mathcal{C}_1| \leq 3^{\mathrm{ex}(n,F_{\ell+1})-1}$. The reason for this small size is, that many pairwise hyperedge-disjoint subhypergraphs F_ℓ (and not $F_{\ell+1}$) arise, and for any such subhypergraph F_ℓ its set of hyperedges cannot be colored completely in green, as otherwise we may build together with the green hyperedges from $\dot\cup_{i=1}^{\ell} E_i(v)$ monochromatic subhypergraphs $F_{\ell+1}$.

Let us make this precise. For every class V_i, $i \in [\ell]$, consecutively we fix exactly $(k-2)$ pairwise distinct vertices $w_1^i, \ldots, w_{k-2}^i$, all distinct from vertex v, such that $\cup_{g<i}\{w_1^g, \ldots, w_{k-2}^g\}$ is disjoint from $\{w_1^i, \ldots, w_{k-2}^i\}$ and which satisfy for n sufficiently large

$$|S_i| = |L_{H_v}(v, w_1^i, \ldots, w_{k-2}^i) \cap V_i| \geq \beta \cdot n/(200 \cdot d_\ell \cdot c_\ell),$$

where $S_i := L_{H_v}(v, w_1^i, \ldots, w_{k-2}^i) \cap V_i$; that is, we concentrate on "green neighborhoods" of vertex v in class V_i, $i \in [\ell]$. Assume in the following for simplicity that $|S_1| = \cdots = |S_\ell| = s := \beta \cdot n/(200 \cdot d_\ell \cdot c_\ell)$ and that s is divisible by $(\ell - 1)$. Consider the on the vertex partition $S_1 \dot\cup \ldots \dot\cup S_\ell$ defined complete ℓ-partite k-uniform hypergraph G. We look for subhypergraphs F_ℓ which are contained in H. Each such subhypergraph F_ℓ, $F_\ell = H_\ell^k$ or $F_\ell = F_\ell^k$, cannot be completely green, as otherwise we obtain in H a green subhypergraph $F_{\ell+1}$ by using the green hyperedges $\{v, w_i, w_1^i, \ldots, w_{k-2}^i\}$, $i \in [\ell]$, where $w_i \in S_i$ is a vertex from the core of F_ℓ.

First we count the number of such hyperedge-disjoint copies of F_ℓ in G (these need not be subhypergraphs in H).

Lemma 5.12. *The induced ℓ-partite k-uniform hypergraph $H[S_1, \ldots, , S_\ell]$ (and therefore the hypergraph H as well) contains at least*

$$c \cdot \beta^k \cdot n^k$$

pairwise hyperedge-disjoint subhypergraphs F_ℓ, where $c > 0$ is a constant depending only on k and ℓ, i.e.

$$c = \frac{\binom{\ell-1}{k-1}}{2 \cdot k \cdot \ell \cdot e(F_\ell) \cdot (200 \cdot d_\ell \cdot c_\ell)^k}.$$

Proof. For finding these subhypergraphs F_ℓ, we use the bounds on the number of hyperedges in different Turán hypergraphs, see (1.8). In particular, $\mathcal{T}_\ell^{(k)}(n)$ is extremal for $F_{\ell+1}$ while $\mathcal{T}_{\ell-1}^{(k)}(n)$ is extremal for F_ℓ when $\ell \geq k+1$. Moreover $\ell = k$, it follows by a result of Erdős [Erd64], that $\mathrm{ex}(n, H_k^k) = o(n^k)$, while $\mathrm{ex}(n, F_k^k) = 1$ is trivial. Thus we have $\mathrm{ex}(\ell\cdot s, F_{\ell+1}) - \mathrm{ex}(\ell\cdot s, F_\ell) = \Theta(s^k)$ for $\ell \geq k$. Namely, with $\mathrm{ex}(\ell\cdot s, F_{\ell+1}) - \mathrm{ex}(\ell\cdot s, F_\ell) > 0$ we know that the complete ℓ-partite k-uniform hypergraph $G = K[S_1, \dots, S_\ell]$ contains at least one copy of F_ℓ, and we remove from G all its $|E(F_\ell)|$ many hyperedges. We may repeat this procedure at least $\xi \cdot s^k$ times, where $\xi = \xi(k, \ell) > 0$ is a constant depending only on k and ℓ for $\ell \geq k$, and this ξ can be computed for $\ell > k$ by the bounds in (1.8), namely

$$\begin{aligned} \mathrm{ex}(\ell \cdot s, F_{\ell+1}) - \mathrm{ex}(\ell \cdot s, F_\ell) &= \binom{\ell}{k} \cdot s^k - \binom{\ell-1}{k} \cdot \left(\frac{\ell \cdot s}{\ell - 1}\right)^k \\ &= \frac{\ell}{k} \cdot \binom{\ell-1}{k-1} \cdot s^k - \frac{\ell-k}{k} \cdot \binom{\ell-1}{k-1} \cdot \left(\frac{\ell \cdot s}{\ell-1}\right)^k \\ &= s^k \cdot \frac{\binom{\ell-1}{k-1}}{k} \cdot \left(\ell - (\ell - k) \cdot \left(\frac{\ell}{\ell-1}\right)^k\right) > s^k \cdot \frac{\binom{\ell-1}{k-1}}{k \cdot \ell}, \end{aligned} \tag{5.69}$$

where the last inequality can be seen as follows: the function $f(k) := (\ell - k)\cdot(\ell/(\ell-1))^k$ is strictly decreasing for $2 \leq k \leq \ell$, which follows from $f(k+1)/f(k) < 1$, hence $\ell - f(k) \geq \ell - f(2) = \ell/(\ell-1)^2 > 1/\ell$.

Having done so, we find in the hypergraph G, assuming that $|S_i| = s$ for every $i \in [\ell]$, at least $\xi \cdot s^k$ hyperedge-disjoint copies of F_ℓ, where by (5.69) it is

$$\xi > \frac{\binom{\ell-1}{k-1}}{k \cdot \ell \cdot e(F_\ell)} \tag{5.70}$$

These $\xi \cdot s^k$ hyperedge-disjoint copies of F_ℓ might not be subhypergraphs F_ℓ in H, as some hyperedges are missing. However, in the hypergraph H less than $\delta \cdot n^k$ crossing hyperedges are missing, hence also in the induced subhypergraph $H[S_1, \dots, , S_\ell]$ of H less than $\delta \cdot n^k$ crossing hyperedges are missing, cf. (5.58). But with $\delta \ll \beta$, see also (5.60), we loose only at most half of the copies F_ℓ already found. Thus, we always find at least

$$\xi \cdot s^k - \delta \cdot n^k \geq (\xi/2) \cdot (\beta \cdot n/(200 \cdot d_\ell \cdot c_\ell))^k \geq c \cdot \beta^k \cdot n^k \tag{5.71}$$

subhypergraphs F_ℓ in H, where $c = (\xi/2) \cdot (1/(200 \cdot d_\ell \cdot c_\ell))^k$. □

Let the pairwise hyperedge-disjoint subhypergraphs of $F_{\ell+1}$ in H be enumerated by $H_1, \dots, H_{c\beta^k n^k}$. Recall that every subhypergraph H_j, $j \in [c\cdot\beta^k\cdot n^k]$, with core $w_1, \dots, w_\ell$, $w_i \in S_i$ for $i \in [\ell]$, together with the hyperedges $\{w_i, v, w_1^i, \dots, w_{k-2}^i\}$, $i \in [\ell]$, builds a subhypergraph $F_{\ell+1}$. Moreover, the latter ℓ hyperedges are all colored the same for *every* coloring $c \in \mathcal{C}_1$. Now, we estimate the number $|\mathcal{C}_1|$ of colorings as follows:

- there are 3 ways to choose the color in which the hyperedges in $\cup_{i=1}^{\ell} E_i(v)$ should be monochromatic, say, in green, and for each class V_i, $i \in [\ell]$, there are at least $\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)$ defective or crossing green hyperedges incident to vertex v that cover class V_i, which yields at most

$$\left(\sum_{i=\beta n^{k-1}/(100 d_\ell c_\ell)}^{\binom{n}{k-1}} \binom{\binom{n}{k-1}}{i} \right)^\ell \leq 2^{\ell n^{k-1}}$$

choices for the green hyperedges in $\cup_{i=1}^{\ell} E_i(v)$, and

- at least $c \cdot \beta^k \cdot n^k$ subhypergraphs F_ℓ together with these green hyperedges yield at least $c \cdot \beta^k \cdot n^k$ copies of $F_{\ell+1}$, and we may therefore color the set of hyperedges in every copy of $F_{\ell+1}$ in at most $(3^{e(F_\ell)}-1)$ instead of $3^{e(F_\ell)}$ ways, as a subhypergraph F_ℓ cannot be monochromatic in green, hence, we consider $e(F_\ell)\cdot c\cdot \beta^k \cdot n^k$ hyperedges, which may be colored in at most

$$(3^{e(F_\ell)} - 1)^{c\beta^k n^k}$$

ways, and

- the set of remaining hyperedges may be colored arbitrarily by 3 colors.

Let $\lambda > 0$, which depends on ℓ only, such that

$$3^{e(F_\ell)-\lambda} = 3^{e(F_\ell)} - 1.$$

With $0 < \delta \leq \lambda \cdot c \cdot \beta^k/2$ and n sufficiently large, we obtain the following upper bound

$$\begin{aligned} |\mathcal{C}_1| &\leq 3 \cdot 2^{\ell n^{k-1}} \cdot 3^{(e(F_\ell)-\lambda)c\beta^k n^k} \cdot 3^{\mathrm{ex}(n,F_{\ell+1})+\delta n^k - e(F_\ell)c\beta^k n^k} \\ &\leq 3 \cdot 2^{\ell n^{k-1}} \cdot 3^{\mathrm{ex}(n,F_{\ell+1})+\delta n^k - c\lambda\beta^k n^k} \leq 3^{\mathrm{ex}(n,F_{\ell+1})-1}. \end{aligned} \tag{5.72}$$

Now we turn to the colorings in $\mathcal{C}_2$ and show that by removing vertex v we obtain for the subhypergraph $H' := H - \{v\}$ on $(n-1)$ vertices the lower bound

$$c_{3,F_{\ell+1}}(H') \geq 3^{\mathrm{ex}(n-1,F_{\ell+1})+m+1}, \tag{5.73}$$

thus, showing the induction hypothesis (5.55).

By (5.72) we already know that

$$|\mathcal{C}_2| = |\mathcal{C}| - |\mathcal{C}_1| \geq 3^{\mathrm{ex}(n,F_{\ell+1})+m} - 3^{\mathrm{ex}(n,F_{\ell+1})-1} \geq 3^{\mathrm{ex}(n,F_{\ell+1})+m-1}. \tag{5.74}$$

Next we estimate the number of 3-colorings in $\mathcal{C}_2$ restricted to the set of all hyperedges incident to vertex v. By (5.59) we know that $|V_i| \leq n/\ell + \ell^2 \cdot \delta^{1/k} \cdot n$, $i \in [\ell]$.

Observe, that in the Turán hypergraph $\mathcal{T}_\ell^{(k)}(n)$ there are only crossing hyperedges, hence in $\mathcal{T}_\ell^{(k)}(n)$ there are at least $\binom{\ell-1}{k-1} \cdot \lfloor n/\ell \rfloor^{k-1}$ hyperedges incident to vertex v. However, in the hypergraph H possibly we have not only crossing types of hyperedges incident to vertex v, i.e., all types may be present. But, as shown in Case 1, the number of bad hyperedges incident to vertex v in H is at most $\beta \cdot n^{k-1}$. In addition to the crossing hyperedges incident to vertex v we possibly have defective hyperedges incident to v, thus, in the worst-case we have to treat the situation that vertex v may be incident to at most

$$\binom{\ell}{k-1} \cdot (n/\ell + \ell^2 \cdot \delta^{1/k} \cdot n)^{k-1} + \beta \cdot n^{k-1}$$

hyperedges. However, our advantage is that we consider colorings from $\mathcal{C}_2$, which imply certain restrictions on the colorings of the hyperedges of these types, and only a certain amount of these hyperedges may be colored arbitrarily by 3 colors.

Let $c \in \mathcal{C}_2$ be a fixed coloring. We know that vertex v covers every class V_i, $i \in [\ell]$ (via some type). On the other hand, for every color col $\in \{\text{green}, \text{blue}, \text{red}\}$, there *must* exist one class $V_{i_{\text{col}}}$, such that whenever v covers V_{col} via some type τ, the number of hyperedges from $E^{\tau}(v)$ colored in col is at most $\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)$. We say such type τ misses color col. Moreover, we say in the case as above that class V_{col} is *missed* by the color col (or the color col *misses* the class V_{col}), i.e., whenever v covers V_{col} by some type τ, τ misses the class V_{col}. Note also, that there are exactly $\binom{\ell-1}{k-2} = \binom{\ell-2}{k-2} + \binom{\ell-2}{k-3}$ defective or crossing types possible incident to vertex v and missing the class V_{col} and the color col. Furthermore notice that if a defective or crossing type (for v) misses some color, then it misses $k-1$ classes, i.e., v covers exactly $k-1$ classes.

We are aiming to show, that the number of colorings in $\mathcal{C}_2$ of the set of hyperedges incident to vertex v is bounded from above by

$$\ell^3 \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{3(c_\ell + d_\ell)} \cdot 2^{A(n/\ell + \ell^2 \delta^{1/k} n)^{k-1}} \cdot 3^{\beta n^{k-1} + B(n/\ell + \ell^2 \delta^{1/k} n)^{k-1}}, \quad (5.75)$$

where A is the number of defective or crossing types that miss exactly one color. Analogously, B is the number of defective or crossing types of hyperedges (incident to vertex v,) that do not miss any color. Notice that we are only interested in those hyperedges and therefore their types that contain vertex V.

Note that our benchmark is the Turán hypergraph $\mathcal{T}_\ell^{(k)}(n)$, where the number of 3-colorings of the set of hyperedges incident to vertex v is precisely

$$3^{\delta(\mathcal{T}_\ell^{(k)}(n))}. \quad (5.76)$$

Our goal is to show that the upper bound (5.75) is much more less than (5.76). Thus, noting that

$$\text{ex}(n-1, F_{\ell+1}) = \text{ex}(n, F_{\ell+1}) - \delta(\mathcal{T}_\ell^{(k)}(n)) = e(\mathcal{T}_\ell^{(k)}(n)) - \delta(\mathcal{T}_\ell^{(k)}(n)), \quad (5.77)$$

concludes the inductive step, i.e., with (5.74) this shows (5.73).

First of all, for a coloring $c \in \mathcal{C}_2$ we note that it is impossible that a class V_i is missed by all colors, as otherwise, vertex v covers class V_i by at most $3 \cdot \beta \cdot n^{k-1}/(100 \cdot c_\ell \cdot d_\ell)$ hyperedges, which contradicts our assumption $|E_{\text{defect}}(v)| \geq \beta \cdot n^{k-1}$ for $i = 1$, or (5.67) for $i \geq 2$. However, the same class may be missed by two of the 3 colors. This case will be considered after the more general one, in which we assume that color col $\in$ {green, red, blue} misses class $V_{i_{\text{col}}}$ with pairwise distinct indices i_{col}.

Distinct Colors Miss Distinct Classes

For convenience, let green miss $V_{i_{\text{green}}}$, red miss $V_{i_{\text{red}}}$ and blue miss $V_{i_{\text{blue}}}$, where i_{green}, i_{red}, i_{blue} are pairwise distinct. Next we calculate, which types of hyperedges can be colored with how many colors.

The number of defective or crossing types of hyperedges, which are incident to vertex v and intersect all classes $V_{i_{\text{green}}}, V_{i_{\text{red}}}$, and $V_{i_{\text{blue}}}$ is exactly $\binom{\ell-3}{k-4}$. Note that $\binom{\ell-3}{i} = 0$ for $i < 0$. Every such type misses *every* class $V_{i_{\text{col}}}$ which it intersects, thus for each color and each such type the number of these hyperedges is less than $\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)$, hence, the number of possible 3-colorings of this set of hyperedges is less than

$$\left(\sum_{i<\beta n^{k-1}/(100 d_\ell c_\ell)} \binom{\binom{n}{k-1}}{i} \right)^{3\binom{\ell-3}{k-4}} \leq \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{3\binom{\ell-3}{k-4}}, \tag{5.78}$$

where we used $\sum_{i=0}^{p} \binom{n}{i} \leq \binom{n}{p+1}$ for $p \leq n/4$ and $n \geq 8$.

The number of defective or crossing types of hyperedges, which are incident to vertex v and intersect exactly two of the classes $V_{i_{\text{green}}}, V_{i_{\text{red}}}$, and $V_{i_{\text{blue}}}$, is $3 \cdot \binom{\ell-3}{k-3}$. In this case, all but less than $2 \cdot \beta \cdot n^k/(100 \cdot d_\ell \cdot c_\ell)$ hyperedges of each of these types can be colored with only one color. Therefore, the number of 3-colorings of this set of hyperedges is at most

$$\left(\sum_{i<\beta n^{k-1}/(100 d_\ell c_\ell)} \binom{\binom{n}{k-1}}{i} \right)^{6\binom{\ell-3}{k-3}} \leq \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{6\binom{\ell-3}{k-3}}. \tag{5.79}$$

The number of defective or crossing types of hyperedges incident to vertex v that intersect exactly one of the classes $V_{i_{\text{green}}}, V_{i_{\text{red}}}$, and $V_{i_{\text{blue}}}$, is $A = 3 \cdot \binom{\ell-3}{k-2}$, where A is the number used in (5.75). Here, for each type, for every involved class $V_{i_{\text{col}}}$, col $\in$ {green, red, blue}, for all but less than $\beta \cdot n^k/(100 \cdot d_\ell \cdot c_\ell)$ hyperedges we can use only 2 colors. This gives at most

$$\begin{aligned} &\left(\sum_{i<\beta n^{k-1}/(100 d_\ell c_\ell)} \binom{\binom{n}{k-1}}{i} \right)^{A} \cdot 2^{A(n/\ell+\ell^2\delta^{1/k}n)^{k-1}} \\ &\qquad \leq \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{3\binom{\ell-3}{k-2}} \cdot 2^{3\cdot\binom{\ell-3}{k-2}(n/\ell+\ell^2\delta^{1/k}n)^{k-1}} \end{aligned} \tag{5.80}$$

3-colorings of this set of hyperedges.

The number of the remaining defective or crossing types of hyperedges incident to vertex v is exactly $B = \binom{\ell-3}{k-1}$, which is our constant B in (5.75). Here we may use all three colors, and combined with the at most $\beta \cdot n^{k-1}$ bad hyperedges incident to vertex

v, this yields at most

$$3^{\beta n^{k-1}+B(n/\ell+\ell^2\delta^{1/k}n)^{k-1}} \tag{5.81}$$

colorings.

By Pascal's identity we have

$$3\cdot\binom{\ell-3}{k-4}+6\cdot\binom{\ell-3}{k-3}+3\cdot\binom{\ell-3}{k-2}=3\cdot\binom{\ell-1}{k-2},$$

and we obtain by (5.78)–(5.81) at most

$$\begin{aligned}&(\ell)_3\binom{\binom{n}{k-1}}{\beta\cdot n^{k-1}/(100\cdot d_\ell\cdot c_\ell)}^{3\binom{\ell-1}{k-2}}\cdot 2^{A(n/\ell+\ell^2\delta^{1/k}n)^{k-1}}\cdot 3^{\beta n^{k-1}+B(n/\ell+\ell^2\delta^{1/k}n)^{k-1}}\\ &\le(\ell)_3\cdot\binom{\binom{n}{k-1}}{\beta\cdot n^{k-1}/(100\cdot d_\ell\cdot c_\ell)}^{3\binom{\ell-1}{k-2}}\cdot\left(2^A\cdot 3^B\right)^{(n/\ell+\ell^2\delta^{1/k}n)^{k-1}}\cdot 3^{\beta n^{k-1}}\end{aligned} \tag{5.82}$$

3-colorings of the set of all hyperedges incident to vertex v.

To see that (5.82) is strictly less than (5.76), we compare two quantities:

$$3^{\binom{\ell-1}{k-1}}\qquad\text{and}\qquad 2^A\cdot 3^B=2^{3\binom{\ell-3}{k-2}}\cdot 3^{\binom{\ell-3}{k-1}}.$$

Let $\zeta>0$ be a constant with $3^{2-\zeta}=2^3$. By Pascal's identity we have

$$\begin{aligned}2^A\cdot 3^B&=2^{3\binom{\ell-3}{k-2}}\cdot 3^{\binom{\ell-3}{k-1}}=3^{2\binom{\ell-3}{k-2}+\binom{\ell-3}{k-1}-\zeta\binom{\ell-3}{k-2}}\\ &=3^{\binom{\ell-2}{k-1}+\binom{\ell-3}{k-2}-\zeta\binom{\ell-3}{k-2}}=3^{\binom{\ell-1}{k-1}-\binom{\ell-2}{k-2}+\binom{\ell-3}{k-2}-\zeta\binom{\ell-3}{k-2}}\\ &=3^{\binom{\ell-1}{k-1}-\binom{\ell-3}{k-3}-\zeta\binom{\ell-3}{k-2}}\end{aligned} \tag{5.83}$$

Thus, with (5.83) and using the entropy function $h(x)$, for $0<\beta,\delta\ll 1$ expression (5.82) can be bounded from above by

$$\begin{aligned}&(\ell)_3\cdot\binom{\binom{n}{k-1}}{\beta\cdot n^{k-1}/(100\cdot d_\ell\cdot c_\ell)}^{3\binom{\ell-1}{k-2}}\cdot\left(2^A\cdot 3^B\right)^{(n/\ell+\ell^2\delta^{1/k}n)^{k-1}}\cdot 3^{\beta n^{k-1}}\\ &\le(\ell)_3\cdot 2^{3\binom{\ell-1}{k-2}h(\beta/(100d_\ell c_\ell))n^{k-1}}\cdot 3^{\beta n^{k-1}+(\binom{\ell-1}{k-1}-\binom{\ell-3}{k-3}-\zeta\binom{\ell-3}{k-2})(1/\ell+\delta^{1/k})^{k-1}n^{k-1}}\\ &\overset{(1.9)}{\le} 3^{\delta(\mathcal{T}_\ell^{(k)}(n))-3},\end{aligned} \tag{5.84}$$

as $h(x)\longrightarrow 0$ with $x\longrightarrow 0$.

Two Colors Miss One Class

Next we consider the case, when two colors miss some class V_i, and, moreover, there exists another class $V_j, j\neq i$, not covered by the third color, i.e., assume that $V_{i_{\text{green}}}=V_{i_{\text{red}}}\neq V_{i_{\text{blue}}}$. We estimate the number of defective or crossing types similarly as above.

The number of defective or crossing types of hyperedges incident to vertex v and intersecting both classes $V_{i_{\mathrm{green}}} = V_{i_{\mathrm{red}}}$ and $V_{i_{\mathrm{blue}}}$ is $\binom{\ell-2}{k-3}$. For each of these types we have for each color less than $\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)$ hyperedges incident to vertex v, thus, the number of 3-colorings of this set of hyperedges is at most

$$\left(\sum_{i<\beta n^{k-1}/(100 d_\ell c_\ell)} \binom{\binom{n}{k-1}}{i} \right)^{3\binom{\ell-2}{k-3}} \leq \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{3\binom{\ell-2}{k-3}}. \tag{5.85}$$

The number of defective or crossing types of hyperedges incident to vertex v and intersecting the class $V_{i_{\mathrm{green}}} = V_{i_{\mathrm{red}}}$ and not intersecting the class $V_{i_{\mathrm{blue}}}$ is $\binom{\ell-2}{k-2}$. For each of these types we have for each of the colors green and red less than $\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)$ hyperedges incident to vertex v and the remaining hyperedges are colored by blue. Thus, the number of 3-colorings of this set of hyperedges is at most

$$\left(\sum_{i<\beta n^{k-1}/(100 d_\ell c_\ell)} \binom{\binom{n}{k-1}}{i} \right)^{2\binom{\ell-2}{k-2}} \leq \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{2\binom{\ell-2}{k-2}}. \tag{5.86}$$

The number of defective or crossing types of hyperedges incident to vertex v, intersecting class $V_{i_{\mathrm{blue}}}$ and disjoint from class $V_{i_{\mathrm{green}}}$ is $A = \binom{\ell-2}{k-2}$, and we can use two colors for coloring all but less than $\beta \cdot n^k/(100 \cdot d_\ell \cdot c_\ell)$ hyperedges incident to vertex v. This gives at most

$$\begin{aligned} &\left(\sum_{i<\beta n^{k-1}/(100 d_\ell c_\ell)} \binom{\binom{n}{k-1}}{i} \right)^{A} \cdot 2^{A(n/\ell + \ell^2 \delta^{1/k} n)^{k-1}} \\ &\leq \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{\binom{\ell-2}{k-2}} \cdot 2^{A(n/\ell + \ell^2 \delta^{1/k} n)^{k-1}} \end{aligned} \tag{5.87}$$

3-colorings of this set of hyperedges.

The number of defective or crossing types of hyperedges incident to vertex v and disjoint from both classes $V_{i_{\mathrm{green}}}$ and $V_{i_{\mathrm{blue}}}$ is $B = \binom{\ell-2}{k-1}$, and we may use all 3 colors for the corresponding hyperedges, i.e., we obtain at most

$$3^{B(n/\ell + \ell^2 \delta^{1/k} n)^{k-1}} \tag{5.88}$$

3-colorings.

Thus, by Pascal's identity we obtain by (5.85)–(5.88), using that there are at most

$\beta \cdot n^{k-1}$ bad hyperedges incident to vertex v, at most

$$
\begin{aligned}
&3(\ell)_2 \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{3\binom{\ell-1}{k-2}} \cdot 2^{A(n/\ell+\ell^2\delta^{1/k}n)^{k-1}} \cdot 3^{\beta n^{k-1}+B(n/\ell+\ell^2\delta^{1/k}n)^{k-1}} \\
&\quad \leq 3(\ell)_2 \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{3\binom{\ell-1}{k-2}} \cdot \left(2^A \cdot 3^B\right)^{(n/\ell+\ell^2\delta^{1/k}n)^{k-1}} \cdot 3^{\beta n^{k-1}} \qquad (5.89)
\end{aligned}
$$

3-colorings of the set of all hyperedges incident to vertex v. To see that (5.89) is less than (5.84), we compare the quantities

$$3^{\binom{\ell-1}{k-1}} \qquad \text{and} \qquad 2^A \cdot 3^B = 2^{\binom{\ell-2}{k-2}} \cdot 3^{\binom{\ell-2}{k-1}}.$$

Let $3^{1-\alpha} = 2$, i.e., $\alpha > 0$ is constant. We infer

$$2^{\binom{\ell-2}{k-2}} \cdot 3^{\binom{\ell-2}{k-1}} = 3^{\binom{\ell-2}{k-2}+\binom{\ell-2}{k-1}-\alpha\binom{\ell-2}{k-2}} = 3^{\binom{\ell-1}{k-1}-\alpha\binom{\ell-2}{k-2}}, \qquad (5.90)$$

hence (5.89) becomes with (5.90):

$$
\begin{aligned}
&3(\ell)_2 \cdot \binom{\binom{n}{k-1}}{\beta \cdot n^{k-1}/(100 \cdot d_\ell \cdot c_\ell)}^{3\binom{\ell-1}{k-2}} \cdot \left(2^A \cdot 3^B\right)^{(n/\ell+\delta^{1/k}n)^{k-1}} \cdot 3^{\beta n^{k-1}} \\
&\leq 3(\ell)_2 \cdot 2^{3\binom{\ell-1}{k-2}h(\beta/(100 d_\ell c_\ell))n^{k-1}} \cdot 3^{\beta n^{k-1}+(\binom{\ell-1}{k-1}-\alpha\binom{\ell-2}{k-2})(1/\ell+\delta^{1/k})^{k-1}n^{k-1}} \qquad (5.91) \\
&\overset{(5.60)}{\leq} 3^{\delta(\mathcal{T}_\ell^{(k)}(n))-3},
\end{aligned}
$$

hence with (5.84) in both situations the number of 3-colorings of the set of hyperedges incident to vertex v is at most

$$3^{\delta(\mathcal{T}_\ell^{(k)}(n))-2},$$

which concludes the inductive hypothesis (5.73) and finishes Case 2.

Case 3: H satisfies $\forall i \in [\ell]$ and $\forall v \in V_i$: $|E_{bad}(v)\dot\cup E_{defect}(v)| \leq 2 \cdot \beta \cdot n^{k-1}$

Here let $F_{\ell+1}$ be $H^k_{\ell+1}$ or $F^k_{\ell+1}$ for $\ell > k \geq 2$. The case $F_{k+1} = H^k_{k+1}$ is very similar to the general one (thus we only sketch it), while for $F_{k+1} = F^k_{k+1}$ a shortcut is necessary, that will be treated at the very end.

Here we are left with the last case, when most of the hyperedges, i.e., at least $\binom{\ell-1}{k-1} \cdot \lfloor n/\ell \rfloor^{k-1} - 2 \cdot \beta \cdot n^{k-1}$ incident to *any* vertex v are crossing.

For a vertex v set $L_{\text{cross}}(v) := \{e \setminus \{v\}|\ v \in e,\ e \in E(H),\ \text{and } e \text{ is crossing}\}$, and for a type τ let $L^\tau(v) := \{e \setminus \{v\} : e \in E(H), e \,\text{has type}\, \tau, v \in e\}$. By assumption $H \neq \mathcal{T}_\ell^{(k)}(n)$, hence there exists a non-crossing hyperedge $e \in E(H)$ with respect to the maximal partition $\mathcal{P}$. Let v_1, v_2 be distinct vertices that belong to this hyperedge e and are contained in the same class, say V_1. As there are at most $2 \cdot \beta \cdot n^{k-1}$ bad or defective hyperedges incident to any vertex v, with (5.59) we know for $0 < \delta \ll \beta$ that

$$\begin{aligned} &|L_{\text{cross}}(v_1) \cap L_{\text{cross}}(v_2)| \\ &\geq 2 \cdot \binom{\ell-1}{k-1} \cdot \left\lfloor \frac{n}{\ell} \right\rfloor^{k-1} - 4 \cdot \beta \cdot n^{k-1} - \binom{\ell-1}{k-1} \cdot \left(\frac{n}{\ell} + \ell^2 \cdot \delta^{1/k} \cdot n \right)^{k-1} \\ &\geq \binom{\ell-1}{k-1} \cdot \left(\frac{n}{\ell}\right)^{k-1} - 5 \cdot \beta \cdot n^{k-1}, \end{aligned} \tag{5.92}$$

recalling the minimum degree condition (5.56) for H. Again by (5.59), we infer for $0 < \delta \ll \beta$, that for *any* crossing type τ in fact, v_1 and v_2 have large common neighborhoods, i.e.,

$$|L^\tau(v_1) \cap L^\tau(v_2)| \geq \left(\frac{n}{\ell}\right)^{k-1} - 6 \cdot \beta \cdot n^{k-1}, \tag{5.93}$$

as otherwise (5.92) is violated.

For crossing types τ let $L^\tau(v_1, v_2) := L^\tau(v_1) \cap L^\tau(v_2)$. Moreover, for a vertex v and some set L of $(k-1)$-element sets let $E(v, L) := \{f \cup \{v\} \mid f \in L\}$.

Again, our argument splits into two cases. We distinguish between two subsets $\mathcal{C}_1$ and $\mathcal{C}_2$ of the set $\mathcal{C}$ of (in Case 3) allowed hyperedge-colorings of H, i.e., $\mathcal{C} = \mathcal{C}_1 \dot\cup \mathcal{C}_2$. Let $\mathcal{C}_1$ be the set of all $F_{\ell+1}$-free 3-colorings of the set of hyperedges of H such that for every crossing type τ that intersects V_1, there exists a subset $L_\tau \subseteq L^\tau(v_1, v_2)$ with $|L_\tau| \geq \varepsilon \cdot (n/\ell)^{k-1}$, for fixed $\varepsilon > 0$, and all hyperedges in $\cup_{\tau \text{ crossing}}(E(v_1, L_\tau) \cup E(v_2, L_\tau))$ and the hyperedge e are colored the same, say in green.

We show first that the size of $\mathcal{C}_1$ is small, i.e., $|\mathcal{C}_1| \leq 3^{\text{ex}(n, F_{\ell+1})-1}$, to finally concentrate on $\mathcal{C}_2$. The reason for the small size of $\mathcal{C}_1$ is, that many pairwise hyperedge-disjoint subhypergraphs $F_{\ell-1}$ (and not F_ℓ) arise, and for each its set of hyperedges cannot be colored completely in green, as otherwise, having such a subhypergraph $F_{\ell-1}$, we may build together with the green hyperedges in $\{e\} \cup \cup_{\tau \text{ cross}}(E(v_1, L_\tau) \cup E(v_2, L_\tau))$ a green subhypergraph $F_{\ell+1}$.

Consider a coloring from $\mathcal{C}_1$. Then, for every crossing type τ of hyperedges containing vertex v_1, there exist subsets $L_\tau \subseteq L^\tau(v_1, v_2)$ with $|L_\tau| = \varepsilon \cdot (n/\ell)^{k-1}$, where

$$0 \ll \delta \ll \beta \ll \varepsilon \ll 1, \tag{5.94}$$

and all k-element sets in $E(v_1, L_\tau) \cup E(v_2, L_\tau)$ are colored green.

For every $j \in \{2, \ldots, \ell\}$, fix a crossing type τ_j intersecting V_1 and V_j.

Fix $j \in [\ell]$. Assume without loss of generality that $j \leq k$ and that the hyperedges of type $\tau := \tau_j$ incident to vertex v_1 intersect each class V_i, $i \in [k]$. Let W_1 and W_2 be the set of all sets $w^1 = \{v_1, w_1^1, \ldots, w_{k-2}^1\}$ and $w^2 = \{v_2, w_1^2, \ldots, w_{k-2}^2\}$, respectively, with $|w^1 \cap V_i| = |w^2 \cap V_i| = 1$, $i \in \{2, \ldots, j-1, j+1, \ldots, k\}$. Let $L_\tau(w^i)$, $i \in [2]$, be the corresponding link-sets (with hyperedges from L_τ only) in class V_j, i.e.,

$$L_\tau(w^i) = \{v \in V_j : (w^i \cup \{v\} \setminus \{v_i\}) \in L_\tau\}.$$

Let $d_\tau(v)$ for $v \in V_j$ be the number of hyperedges in $\{\{v_1\} \cup f\colon\ f \in L_\tau\}$ incident to vertex v.

Then, for $0 < \delta < (1/\ell)^{3k}$ we infer by using the Cauchy-Schwarz inequality

$$\begin{aligned}\sum_{(w^1,w^2)\in W_1\times W_2} |L_\tau(w^1) \cap L_\tau(w^2)| &= \sum_{v\in V_j} (d_\tau(v))^2 \\ \geq \frac{\left(\sum_{v\in V_j} d_\tau(v)\right)^2}{|V_j|} = |L_\tau|^2/|V_j| &\geq \frac{\left(\varepsilon\cdot\left(\frac{n}{\ell}\right)^{k-1}\right)^2}{\frac{n}{\ell}+\ell^2\cdot\delta^{1/k}\cdot n} \geq \frac{\varepsilon^2}{2}\cdot\left(\frac{n}{\ell}\right)^{2k-3}.\end{aligned} \tag{5.95}$$

The number of pairs $(w^1, w^2) \in W_1 \times W_2$, which have at least one common entry, is less than

$$k\cdot\left(\frac{n}{\ell}+\ell^2\cdot\delta^{1/k}\cdot n\right)^{2k-5},$$

and their contribution to the first sum in (5.95) is at most

$$k\cdot\left(\frac{n}{\ell}+\ell^2\cdot\delta^{1/k}\cdot n\right)^{2k-4} = o(n^{2k-3}),$$

hence we know by (5.95) that for n sufficiently large

$$\sum_{(w^1,w^2)\in W_1\times W_2;\, w^1\cap w^2=\varnothing} |L_\tau(w^1) \cap L_\tau(w^2)| \geq \frac{\varepsilon^2}{4}\cdot\left(\frac{n}{\ell}\right)^{2k-3}.$$

This implies that for $0 < \delta < ((2^{1/(2k-4)}-1)/\ell^3)^k$ there exists a pair $(w^1, w^2) \in W_1 \times W_2$ with $w^1 \cap w^2 = \varnothing$ and we have

$$|L_\tau(w^1) \cap L_\tau(w^2)| \geq \frac{\frac{\varepsilon^2}{4}\cdot\left(\frac{n}{\ell}\right)^{2k-3}}{\left(\frac{n}{\ell}+\ell^2\cdot\delta^{1/k}\right)^{2k-4}} \geq \frac{\varepsilon^2}{8}\cdot\frac{n}{\ell}.$$

Doing this for every $j \in \{2,\dots,\ell\}$, by the above averaging argument and (5.93), using our assumption $|L_\tau| = \varepsilon\cdot(n/\ell)^{k-1}$ for every crossing type τ, for every class V_j, $j \in \{2,\dots \ell\}$, there is a crossing type τ_j of hyperedges containing vertex v_1 (and similarly containing vertex v_2), such that for $i \in [2]$ consecutively we may find $(k-2)$ distinct vertices $w_1^{(i,j)},\dots,w_{k-2}^{(i,j)}$ with $\{w_1^{(i,j)},\dots,w_{k-2}^{(i,j)}\}$ disjoint from $\{v_1, v_2\}$ and the hyperedge e, and also

$$\{w_1^{(i,j)},\dots,w_{k-2}^{(i,j)}\} \cap \{w_1^{(i',j')},\dots,w_{k-2}^{(i',j')}\} = \varnothing,$$

whenever $(i,j) \neq (i',j')$, and there exist subsets $S_j \subset V_j$ with

$$|S_j| \geq \varepsilon^2\cdot n/(10\cdot\ell),$$

such that for all $i \in [2]$, and $j \in \{2,\dots,\ell\}$, and $s_j \in S_j$ we have

$$\{v_i, w_1^{(i,j)},\dots,w_{k-2}^{(i,j)}, s_j\} \in E(v_i, L_{\tau_j}). \tag{5.96}$$

Moreover, all sets S_j, $j \in [\ell]$, are disjoint from any set $\{w_1^{(i',j')}, \ldots, w_{k-2}^{(i',j')}\}$ and from the hyperedge e.

By possibly omitting some vertices, we assume that $|S_2| = \cdots = |S_\ell| = s := \varepsilon^2 \cdot n/(10 \cdot \ell)$.

For $\ell > k$, we consider the complete $\ell - 1$-partite k-uniform hypergraph G with vertex partition $S_2 \dot\cup \ldots \dot\cup S_\ell$. No subhypergraph $F_{\ell-1}$ in G with core $s_2, \ldots, s_\ell$, $s_i \in S_i$, can be colored completely in green, as otherwise we obtain a green subhypergraph $F_{\ell+1}$ in H by using the hyperedge e and the hyperedges $\{v_i, w_1^{(i,j)}, \ldots, w_{k-2}^{(i,j)}, s_j\}$, $s_j \in S_j$, $i \in [2]$ and $j \in \{2, \ldots, \ell\}$.

By Lemma 5.12, cf. (5.69)-(5.71), for $\ell > k$ there exist $\xi \cdot s^k$ with

$$\xi > \frac{\binom{\ell-2}{k-1}}{2 \cdot k \cdot (\ell - 1) \cdot e(F_{\ell-1})}$$

hyperedge-disjoint copies of $F_{\ell-1}$ in G.

These copies of $F_{\ell-1}$ might not be subhypergraphs $F_{\ell-1}$ in $H[S_2, \ldots, S_\ell]$. However, in the hypergraph H hence also in the subhypergraph $H[S_2, \ldots, S_\ell]$ less than $\delta \cdot n^k$ crossing hyperedges are missing, cf. (5.58). With $\delta \le (\xi/2) \cdot (\varepsilon^2/(10 \cdot \ell))^k$, we always find at least

$$\xi \cdot s^k - \delta \cdot n^k = \xi \cdot (\varepsilon^2 \cdot n/(10 \cdot \ell))^k - \delta \cdot n^k \overset{(5.94)}{\ge} (\xi/2) \cdot (\varepsilon^2 \cdot n/(10 \cdot \ell))^k \ge c \cdot \varepsilon^{2k} \cdot n^k \quad (5.97)$$

subhypergraphs $F_{\ell-1}$ in H, where $c = \xi/(2 \cdot 10^k \cdot \ell^k)$. Let the hyperedge-disjoint subhypergraphs of $F_{\ell-1}$ in G be enumerated by $H_1, \ldots, H_{c\varepsilon^{2k} n^k}$.

Every subhypergraph H_j, $j \in [c \cdot \varepsilon^{2k} \cdot n^k]$, together with the hyperedge e and the hyperedges $\{s_j, v_i, w_1^{(i,j)}, \ldots, w_{k-2}^{(i,j)}\}$, $i \in [2]$ and $j \in \{2, \ldots, \ell\}$, and vertices $s_j \in S_j$ from the core of $F_{\ell-1}$ yields a subhypergraph $F_{\ell+1}$. Moreover, the latter hyperedges are all colored the same for *every* coloring $c \in \mathcal{C}_1$.

Now, we estimate the cardinality of $\mathcal{C}_1$ as follows:

- there are 3 choices for the color in which the $(2 \cdot \ell - 1)$ hyperedges should be colored, say, in green, and for every class V_j, $j \in \{2, \ldots, \ell\}$, there are at least $\varepsilon \cdot (n/\ell)^{k-1}$ green hyperedges that cover class V_j, which yields at most

$$\left(\sum_{i=\varepsilon(n/\ell)^{k-1}}^{\binom{n}{k-1}} \binom{\binom{n}{k-1}}{i} \right)^{\ell-1} \le 2^{\binom{n}{k-1}(\ell-1)} \le 2^{\ell n^{k-1}}$$

  choices for these green hyperedges, and

- at least $c \cdot \varepsilon^k \cdot n^k$ subhypergraphs $F_{\ell-1}$ together with at most $(2 \cdot \ell - 1)$ green hyperedges yield at least $c \cdot \varepsilon^{2k} \cdot n^k$ subhypergraphs $F_{\ell+1}$, and we may color the set of hyperedges in every subhypergraph $F_{\ell-1}$ in at most $(3^{e(F_{\ell-1})} - 1)$ instead of $3^{e(F_{\ell-1})}$ ways, as a subhypergraph $F_{\ell-1}$ cannot be monochromatic in green, hence in total, we consider $e(F_{\ell-1}) \cdot c \cdot \varepsilon^{2k} \cdot n^k$ many hyperedges, which may be colored in at most

$$(3^{e(F_{\ell-1})} - 1)^{c\varepsilon^{2k}n^k}$$

ways, and

- the remaining hyperedges may be colored arbitrarily by 3 colors.

Let $\lambda > 0$ be such that

$$3^{e(F_{\ell-1})-\lambda} = 3^{e(F_{\ell-1})} - 1.$$

With $0 < \delta < \lambda \cdot c \cdot \varepsilon^{2k}/2$, for n sufficiently large, we obtain the following upper bound

$$\begin{aligned}|\mathcal{C}_1| &\leq 3 \cdot 2^{\ell n^{k-1}} \cdot 3^{e(F_{\ell-1})-\lambda)c\varepsilon^{2k}n^k} \cdot 3^{\mathrm{ex}(n,F_{\ell+1})-e(F_{\ell-1})c\varepsilon^{2k}n^k+\delta n^k} \\ &\leq 3 \cdot 2^{\ell n^{k-1}} \cdot 3^{\mathrm{ex}(n,F_{\ell+1})+\delta n^k - c\lambda\varepsilon^{2k}n^k} \\ &\overset{(5.94)}{\leq} 3^{\mathrm{ex}(n,F_{\ell+1})-1}.\end{aligned} \tag{5.98}$$

Next, we explain how to adjust the arguments to the case when $F_{k+1} = H^k_{k+1}$. There we need an additional vertex set $S_1 \subset V_1$, which does not contain any vertices $(v_i, w_t^{(i,j)})$ previously chosen. Now we clearly find $\Theta(n^k)$, see [Erd64]– H^k_k is k-partite k-uniform hypergraph, many hyperedge disjoint copies of H^k_k (and therefore of H^k_{k-1}) in $H[S_1, \ldots, S_\ell]$. The remaining argument goes analogously.

Next we turn to the colorings in $\mathcal{C}_2 = \mathcal{C} \setminus \mathcal{C}_1$. By (5.98) we know for $\ell \geq k$ that

$$|\mathcal{C}_2| = |\mathcal{C}| - |\mathcal{C}_1| \geq 3^{\mathrm{ex}(n,F_{\ell+1})+m} - 3^{\mathrm{ex}(n,F_{\ell+1})-1} \geq 3^{\mathrm{ex}(n,F_{\ell+1})+m-1}. \tag{5.99}$$

Fix a coloring from $\mathcal{C}_2$. By (5.59), for each crossing type τ there are at most $(n/\ell + \ell^2 \cdot \delta^{1/k} \cdot n)^{k-1}$ hyperedges incident to any fixed vertex v. As we consider colorings from $\mathcal{C}_2$, there must exist a crossing type τ such that we have less than $\varepsilon \cdot (n/\ell)^{k-1}$ hyperedges $f \in E(H)$ incident to vertex v_1, where $f \setminus \{v_1\}$ is contained in $L^\tau(v_1, v_2)$, which have the same color, say green, as the hyperedge e. Let L be this set of $(k-1)$-element subsets $f \setminus \{v_1\}$, $f \in E(H)$. These hyperedges can be chosen in at most

$$\sum_{i<\varepsilon\cdot(n/\ell)^{k-1}} \binom{\binom{n}{k-1}}{i} \leq \binom{\binom{n}{k-1}}{\varepsilon \cdot (n/\ell)^{k-1}} \leq 2^{h(\varepsilon)n^{k-1}}$$

ways.

Thus, with (5.93) for this type τ we can color all but at most $2 \cdot \varepsilon \cdot (n/\ell)^{k-1} + 6 \cdot \beta \cdot n^{k-1}$ hyperedges from the set $E^\tau(v_1) \cup E^\tau(v_2)$ in at most 8 instead of 9 ways, as for every $(k-1)$-element set f from $L^\tau(v_1, v_2) \setminus L$ we cannot color both hyperedges $\{v_1\} \cup f$ and $\{v_2\} \cup f$ in green.

There are at most

$$\left(\binom{\ell-1}{k-1} - 1\right) \cdot \left(\frac{n}{\ell} + \ell^2 \cdot \delta^{1/k} \cdot n\right)^{k-1}$$

crossing hyperedges incident to vertex v_1 of a type distinct from τ, which may be colored by at most 3 colors, hence for this set of hyperedges we obtain at most

$$3^{(\binom{\ell-1}{k-1}-1)(n/\ell+\ell^2\delta^{1/k}n)^{k-1}}$$

3-colorings, and similarly for vertex v_2.

There are 3 choices for the color of the hyperedge e, hence altogether, as there are at most $4\cdot\beta\cdot n^{k-1}$ bad or defective hyperedges incident to vertex v_1 or v_2, we have at most

$$3\binom{\ell-1}{k-1}\cdot 2^{h(\varepsilon)n^{k-1}}8^{(n/\ell+\ell^2\delta^{1/k}n)^{k-1}}\cdot 3^{2\left(2\beta n^{k-1}+(\binom{\ell-1}{k-1}-1)(n/\ell+\ell^2\delta^{1/k}n)^{k-1}\right)} \tag{5.100}$$

different 3-colorings of hyperedges that contain either v_1 or v_2 (or both).. For $0<\delta\ll\beta\ll\varepsilon\ll 1$ the upper bound (5.100) is less than

$$3^{\delta(\mathcal{T}_\ell^{(k)}(n))+\delta(\mathcal{T}_\ell^{(k)}(n-1))-2}. \tag{5.101}$$

Hence we can delete both vertices v_1 and v_2 and consider the subhypergraph $H':=H-\{v_1,v_2\}$. For $\ell>k$, with (5.77) and (5.99) the induction step is finished and yields

$$c_{3,F_{\ell+1}}(H')\geq\frac{|\mathcal{C}_2|}{3^{\delta(\mathcal{T}_\ell^{(k)}(n))+\delta(\mathcal{T}_\ell^{(k)}(n-1))-2}}\geq 3^{\mathrm{ex}(n-2,f_{\ell+1})+m+1},$$

which proves (5.55).

Finally, we turn to the case when $F_{k+1}=F^k_{k+1}$. We again assume that a non-crossing hyperedge e intersects V_1 in some vertices v_1,v_2. We consider then the $(k-1)$-uniform hypergraph $G:=L_{\mathrm{cross}}(v_1)\cap L_{\mathrm{cross}}(v_2)\cap[V(H)\setminus(e\cup V_1)]^{k-1}$ (notice that crossing hyperedges are with respect to the vertex partition $\mathcal{P}$ and here we identify the hypergraph G with the set of its hyperedges). Using (5.93) we know

$$e(H')\geq\left(\frac{n}{k}\right)^{k-1}-6\beta n^{k-1}$$

and therefore G contains at least

$$\frac{1}{k}\cdot\left(\frac{n}{k}\right)^{k-1}-7\beta n^{k-1}$$

hyperedge-disjoint copies of the following $(k-1)$partite $(k-1)$-uniform hypergraph F' with the vertex set

$$V(F')=\{1,\dots,(k-1)^2\}$$

and the hyperedge set

$$E(F')=\{\{1,\dots,k-1\},\dots,\{(k-2)(k-1)+1,\dots,(k-1)^2\},\{1,k,\dots,(k-2)(k-1)+1\}\}.$$

Clearly, a copy of F' in G will form together with the hyperedge e two copies of F^k_{k+1} that share only the hyperedge e. Thus we find in this way at least

$$\frac{2}{k} \cdot \left(\frac{n}{k}\right)^{k-1} - 14\beta n^{k-1}$$

copies of F^k_{k+1} in H that all have only one hyperedge e in common. This means that having fixed the color of e (3 possibilities), we can color any of the above copies of F^k_{k+1} in at most $3^k - 1$ instead of 3^k many ways. Therefore, we upper bound the number of colorings of the hyperedges that are incident to either v_1 or v_2 by

$$\begin{aligned} 3 \cdot (3^k - 1)^{(2/k)\cdot(n/k)^{k-1} - 14\beta n^{k-1}} 3^{2(n/k + k^2\delta^{1/k} n)^{k-1} + 4\beta n^{k-1} + n^{k-2} - (2/k)\cdot(n/k)^{k-1} + 14\beta n^{k-1}} \\ \le 3^{\delta(\mathcal{T}^{(k)}_\ell(n)) + \delta(\mathcal{T}^{(k)}_\ell(n-1)) - 2}. \end{aligned} \tag{5.102}$$

Thus, deleting v_1 and v_2 from H implies for $H' := H \setminus \{v_1, v_2\}$ with (5.102) that

$$c_{3,F^k_{k+1}}(H') \ge 3^{\mathrm{ex}(n-2, F^k_{k+1}) + m + 1},$$

and this finishes the Claim 3 and proves Lemma 5.6. □

5.3 Using more than 3 colors

Proof of Theorem 1.15.

Fano plane

Let $H = (V, E)$ be the complete 4-partite hypergraph with the vertex partition $V = V_1 \dot\cup V_2 \dot\cup V_3 \dot\cup V_4$ of almost equal size: $||V_i| - |V_j|| \le 1$ for $1 \le i < j \le 4$. We color its hyperedges with colors from $[r]$ as follows. The hyperedges from $E(V_1 \cup V_3, V_2 \cup V_4)$ can be colored with colors from $\{1, \dots, r-2\}$, from $E(V_1 \cup V_2, V_3 \cup V_4)$ with color $r-1$ and from $E(V_1 \cup V_4, V_2 \cup V_3)$ with color r. Obviously, there are no monochromatic Fano planes, as all monochromatic induced subhypergraphs are bipartite. It remains to verify a lower bound on the number of possible colorings (we now assume for simplicity that 4 divides n):

- the hyperedges that intersect 3 of the possible 4 partition classes can be colored arbitrarily (i.e., by r colors), which gives

  $$r^{4(\frac{n}{4})^3}$$

  colorings for those hyperedges,

- the hyperedges from $E(V_1, V_2)$, $E(V_1, V_4)$, $E(V_2, V_3)$ or $E(V_3, V_4)$ can be colored with $r-1$ colors and since $e(V_i, V_j) = 2\binom{n/4}{2}\frac{n}{4}$ we obtain:

  $$(r-1)^{4\cdot 2\binom{n/4}{2}\frac{n}{4}}$$

colorings for these hyperedges,

- the hyperedges from $E(V_1, V_3)$ or $E(V_2, V_4)$ can be colored with 2 colors in

$$2^{2\cdot 2\binom{n/4}{2}\frac{n}{4}}$$

many ways.

Consequently,

$$c_{4,F}(n) \geq r^{4(\frac{n}{4})^3}(r-1)^{4\cdot 2\binom{n/4}{2}\frac{n}{4}} 2^{2\cdot 2\binom{n/4}{2}\frac{n}{4}} \geq \left(\sqrt{\sqrt{2}r(r-1)}\right)^{n^3/8-O(n^2)} \geq (r+\varepsilon)^{e(B_n)}$$

for any $r \geq 4$ and for some $\varepsilon > 0$ and sufficiently large n.

We note that this lower bound on the number of Fano plane-free r-colorings can be easily improved. For example, if one distributes the available colors for the three bipartitions as evenly as possible, then one obtains the following for $r \geq 4$

$$c_{r,F}(n) \geq f_r^{n^3/8-O(n^2)}, \text{with } f_r = \begin{cases} \left(\frac{2}{3}\right)^{3/4} r^{5/4} & \text{if } r = 0 \mod 3 \\ r^{1/2}\left\lceil \frac{2}{3}r \right\rceil^{1/2} \left\lfloor \frac{2}{3}r \right\rfloor^{1/4} & \text{if } r = 1 \mod 3 \\ r^{1/2}\left\lceil \frac{2}{3}r \right\rceil^{1/4} \left\lfloor \frac{2}{3}r \right\rfloor^{1/2} & \text{if } r = 2 \mod 3. \end{cases}$$

Generalized triangles T_3 and T_4

Below we prove lower bounds $c_{r,T_3}(n) \gg r^{\mathrm{ex}(n,T_3)}$ and $c_{r,T_4}(n) \gg r^{\mathrm{ex}(n,T_4)}$ for $r \geq 4$. In the following we assume for simplicity that n is divisible by 3 and 4.

First we consider the case of the 3-uniform generalized triangle T_3. To prove a lower bound on $c_{r,T_3}(n)$ we give a lower bound on $c_{r,K_3}(n)$, i.e., for the case of graphs, where we forbid a monochromatic triangle, see also [ABKS04]. Namely, consider the following graph $G = (V, E)$ with $|V| = n$ vertices. Let $V = V_1 \dot\cup V_2 \dot\cup V_3 \dot\cup V_4$ be a partition of the vertex set V with $|V_i| = n/4$, $i \in [4]$. The edge set E of G consists of all edges $e = \{v, w\}$ with $v \in V_i$ and $w \in V_j$, where $i \neq j$. Given the set $[r]$, $r \geq 4$, of colors, we color the set of all edges between classes V_1 and V_2, or between V_3 and V_4 by the colors $1, \ldots, r-1$. For the set of all edges between the classes V_1 and V_4, or V_2 and V_3 we use the colors $1, \ldots, r-2, r$. Moreover, the set of all edges between the classes V_1 and V_3, or V_2 and V_4 are colored arbitrarily by the colors $r-1$ and r. Here every coloring gives rise to a monochromatic bipartite graph, so no monochromatic triangle is created by the colorings described above.

The number of these colorings in G for $r \geq 4$ is

$$c_{r,K_3}(n) \geq c_{r,K_3}(G) = (r-1)^{4\left(\frac{n}{4}\right)^2} \cdot 2^{2\left(\frac{n}{4}\right)^2} = \left((r-1)\cdot\sqrt{2}\right)^{\frac{n^2}{4}} \gg r^{\frac{n^2}{4}} \geq r^{\mathrm{ex}(n,K_3)}. \tag{5.103}$$

The lower bound (5.103) may be improved by using another distribution of the set $[r]$ of colors, namely for r divisible by 3 say, we color the set of all edges between the classes

V_1 and V_2, or V_3 and V_4 by the colors $1, \dots, 2r/3$. For the set of all edges between the classes V_1 and V_4, or V_2 and V_3 we use the colors $1, \dots, r/3, 2r/3+1, \dots, r$. Moreover, the set of all edges between the classes V_1 and V_3, or V_2 and V_4 are colored arbitrarily by the colors $r/3+1, \dots, r$, which gives

$$c_{r,K_3}(n) \geq \left(\left(\frac{2r}{3} \right)^{\frac{3}{2}} \right)^{\frac{n^2}{4}} \gg r^{\mathrm{ex}(n,K_3)} \tag{5.104}$$

colorings.

Now we consider the 3-uniform generalized triangle T_3 and the 3-uniform, 2-partite hypergraph $H_3 = (V, E)$ on $|V| = n$ vertices, which is defined as follows. Let $V = V_0 \dot{\cup} V'$ be a partition with $|V_0| = n/3$ and $|V'| = 2n/3$. All hyperedges $e \in E$ contain exactly one vertex from V_0 and two vertices from V'. On the set V' we place the graph G from above with $m = 2n/3$ vertices. For any hyperedge $e = \{v_0, v, w\} \in E$ with $e \cap V_0 = \{v_0\}$ its link $\{v, w\}$ has to be an edge in the graph G. The hyperedge $e = \{v_0, v, w\}$ may be colored by some color by which the edge $\{v, w\}$ may be colored.

Using (5.103), this yields

$$\begin{aligned} c_{r,T_3}(n) \geq c_{r,T_3}(H_3) = \left(\left((r-1) \cdot \sqrt{2} \right)^{\frac{(2n/3)^2}{4}} \right)^{\frac{n}{3}} &= \left((r-1) \cdot \sqrt{2} \right)^{\frac{n^3}{27}} \\ &\gg r^{\frac{n^3}{27}} \geq r^{\mathrm{ex}(n,T_3)} \end{aligned} \tag{5.105}$$

colorings for $r \geq 4$ and n sufficiently large. Of course, (5.105) may be improved by using (5.104).

It remains to show that the hypergraph H_3 does not contain a generalized triangle T_3. If $\{a, b, c\}$, $\{b, c, d\}$ and $\{a, d, e\}$ is a subhypergraph T_3 in H_3, then one of the two vertices b or c, and e must be contained in class V_0, say $b, e \in V_0$. But then the union of the links of the vertices b and d forms a triangle in the graph G. However, due to the construction of the colorings, there is no monochromatic triangle T_2 in G, hence no monochromatic triangle T_3.

Next we consider the 4-uniform generalized triangle T_4 and the 4-uniform, 2-partite hypergraph $H_4 = (V, E)$ on $|V| = n$ vertices, which is defined as follows. Let $V = V_0 \dot{\cup} V'$ be a partition with $|V_0| = n/4$ and $|V'| = 3n/4$. All hyperedges $e \in E$ contain exactly one vertex from V_0 and three vertices from V'. On the set V' we place the hypergraph H_3 from above with $m = 3n/4$ vertices. For any hyperedge $e = \{v_0, v, w, x\} \in E$ with $e \cap V_0 = \{v_0\}$ its link $\{v, w, x\}$ has to be a hyperedge in the hypergraph H_3. The hyperedge $e = \{v_0, v, w, x\}$ may be colored by some color by which the hyperedge $\{v, w, x\}$ in H_3 may be colored.

With (5.105), this gives

$$c_{r,T_4}(n) \geq c_{r,T_4}(H_4) = \left(\left((r-1)\cdot\sqrt{2} \right)^{\frac{(3n/4)^3}{27}} \right)^{\frac{n}{4}} = \left((r-1)\cdot\sqrt{2} \right)^{\frac{n^4}{256}}$$
$$\gg r^{\frac{n^4}{256}} \geq r^{\mathrm{ex}(n,T_4)}$$

colorings for $r \geq 4$ and n sufficiently large.

It remains to show that the hypergraph H_4 does not contain a generalized triangle T_4. If $\{a,b,c,d\}$, $\{e,b,c,d\}$ and $\{a,e,f,g\}$ is a subhypergraph T_4 in H_4, then one of the three vertices b, c or d, and f or g must be contained in class V_0, say $b, f \in V_0$. But then the union of the links of b and f forms a generalized triangle in the hypergraph H_3. However, due to the construction of the colorings, there is no monochromatic generalized triangle T_3, hence no monochromatic generalized triangle T_4.

Expanded complete graph and Fan(k)-hypergraph
We show for fixed $r \geq 4$ for the expanded complete graph $H^k_{\ell+1}$ and for the Fan(k) hypergraph $F^k_{\ell+1}$ the lower bound which is exponentially larger than $r^{e(\mathcal{T}^{(k)}_\ell(n))}$ for n sufficiently large.

Let V be an n-element vertex set and we assume for simplicity that 2ℓ divides n. Consider a partition $\mathcal{P}$ of the vertex set V into $(\ell+2)$ pairwise disjoint vertex sets $V_1,\ldots,V_{\ell-2},W_1,\ldots,W_4$, where each class V_i, $i \in [\ell-2]$, has cardinality $|V_i| = n/\ell$, and every other class W_i, $i \in [4]$, satisfies $|W_i| = n/(2\cdot\ell)$. Let H be the k-uniform $(\ell+2)$-partite hypergraph with respect to the partition $\mathcal{P}$, where all crossing hyperedges are present except for those that intersect more than two classes $W_i, W_j, i \neq j$. Let $\{1,\ldots,r\}$ be the set of colors.

All hyperedges in $E(H)$ which contain at most one vertex from $W_1 \cup \cdots \cup W_4$ can be colored with all r colors. All hyperedges in $E(H)$ which contain one vertex from each class W_1 and W_2 or from each class W_3 and W_4 are colored with $1,\ldots,r-1$. All hyperedges in $E(H)$ which contain one vertex from each class W_1 and W_3 or from each class W_2 and W_4 get colors $1,\ldots,r-2,r$. All hyperedges in $E(H)$ which contain one vertex from each class W_1 and W_4 or from each class W_2 and W_3 are colored with $r-1,r$. Note that the projection (link) of any three hyperedges on $W_1 \cup \cdots \cup W_4$ does not give a monochromatic graph triangle.

Then, the number of colorings of the set $E(H)$ of hyperedges of H is precisely

$$r^{\binom{\ell-2}{k}(n/\ell)^k + 2\binom{\ell-2}{k-1}(n/\ell)^k} \cdot (r-1)^{\binom{\ell-2}{k-2}(n/\ell)^k} \cdot 2^{(1/2)\binom{\ell-2}{k-2}(n/\ell)^k}$$
$$= \left(r^{\binom{\ell}{k}} \cdot \frac{\left((r-1)\cdot\sqrt{2}\right)^{\binom{\ell-2}{k-2}}}{r^{\binom{\ell-2}{k-2}}} \right)^{(n/\ell)^k}$$
$$\gg r^{\binom{\ell}{k}(n/\ell)^k} \geq r^{\mathrm{ex}(n,H^k_{\ell+1})}$$

for n sufficiently large.

Suppose for contradiction that for one of these colorings the hypergraph H contains a monochromatic $H^k_{\ell+1}$ with core $v_1, \ldots, v_{\ell+1}$. As all hyperedges in H are crossing, at least three vertices of the core of $H^k_{\ell+1}$ must be contained in the set $W_1 \cup \cdots \cup W_4$. Without loss of generality let v_1, v_2, v_3 be such vertices. By construction, no two of these can be contained in the same vertex set W_i. For each pair $\{v_i, v_j\}$, $1 \leq i < j \leq 3$, there is a $(k-2)$-element set $S_{i,j}$ such that $\{v_i, v_j\} \cup S_{i,j}$ is a hyperedge in $H^k_{\ell+1}$, and again by construction $S_{i,j} \subseteq V_1 \cup \cdots \cup V_{\ell-2}$, but then the links $L_H(S_{i,j})$, $1 \leq i < j \leq 3$, yield a graph triangle in $W_1 \cup \cdots \cup W_4$, hence the hyperedges $\{v_i, v_j\} \cup S_{i,j}$, $1 \leq i < j \leq 3$, do not have all the same color.

Now assume that we obtain a monochromatic subhypergraph $F^k_{\ell+1}$ with core vertices $v_1, \ldots, v_{\ell+1}$ where $v_1, \ldots, v_k$ form a hyperedge in $F^k_{\ell+1}$. Then at least three of the core vertices must be contained in the set $W_1 \cup \cdots \cup W_4$, say these are the vertices v_g, v_h, v_i, where $1 \leq g < h < i$. We must have $g \leq k$ as otherwise we proceed similarly to the paragraph above to obtain a contradiction. Moreover, by construction we cannot have $i \leq k$, as $v_1, \ldots, v_k$ form a hyperedge of $F^k_{\ell+1}$. If $g, h \leq k$, then with the sets $S_{g,i}$ and $S_{h,i}$ forming a hyperege with $\{g, i\}$ and $\{h, i\}$, respectively, the links $L(\{v_1, \ldots, v_k\} \backslash \{v_g, v_h\})$, $L(S_{g,i})$, and $L(S_{h,i})$ yield a monochromatic triangle in $W_1 \cup \cdots \cup W_4$, which is not possible. On the other hand, if $h \geq k+1$, the same reasoning applies, and we are finished.

The lower bound can be improved for larger values of r by better distributing the colors, similarly to (5.104), which gives (for r divisible by 3) the lower bound

$$c_{r,H^k_{\ell+1}}(n), c_{r,F^k_{\ell+1}}(n) \geq \left(r^{\binom{\ell}{k}} \cdot \left(\frac{2 \cdot \sqrt{2r}}{3\sqrt{3}} \right)^{\binom{\ell-2}{k-2}} \right)^{(n/\ell)^k}. \tag{5.106}$$

□

5.4 Upper Bounds on $c_{r,F}(n)$ for $r \geq 4$

The next result, Theorem 1.16 gives an upper bound on $c_{r,F}$ when $r \geq 4$ and F is a k-uniform hypergraph. Unfortunately, the upper bound we achieve is exponentially far away from the lower bounds of the constructions for particular hypergraphs that we presented in the previous section.

Proof of Theorem 1.16. The arguments are similar to those used in the proof of Theorem 1.11. In fact, we repeat the proof of Theorem 1.11 till the case analysis therein almost verbatim.

Given $\varepsilon > 0$. We choose positive ξ such that

$$\frac{h(2k!r\xi)}{k!} \leq \varepsilon/4 \qquad \text{and} \qquad 2r\xi\ln(r) \leq \varepsilon/4. \tag{5.107}$$

Now apply counting lemma, Theorem 2.17, with F and $d_k = \xi$ obtaining $\delta_k > 0$. We may assume that

$$\delta_k \leq \xi/2, \tag{5.108}$$

as setting δ_k smaller makes the complexes we consider more regular (and therefore the statements still hold). We choose $\eta > 0$ as follows

$$\eta \leq \xi/2, \tag{5.109}$$

so that for every $a \geq 1/(2\eta)$, if a hypergraph on a vertices has at least

$$\pi_F \cdot e^{\varepsilon/5}\binom{a}{k} \tag{5.110}$$

hyperedges, then it contains a copy of F. Note that because $\mathrm{ex}(n, F)/\binom{n}{k}$ is a monotone decreasing function, which converges to π_F, such a choice is always possible. Recall also that $a_1 \geq (1/2\eta)$ for an equitable family of partitions (cf. property (a) of Definition 2.13).

Let $\delta\colon \mathbb{N}^{k-1} \to (0, 1]$ and $f\colon \mathbb{N}^{k-1} \to \mathbb{N}$ be the functions guaranteed by Theorem 2.17. Also, we require, that the number of cliques of size k that are spanned by *any* $(\delta(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{d})$-regular $(n/a_1, k, k-1)$-complex should lie in the range

$$(1 \pm \varepsilon/4)\left(\frac{n}{a_1}\right)^k \Big/ \prod_{i=2}^{k-1} a_i^{\binom{k}{i}}.$$

The existence of an appropriately small function δ is asserted by [KRS02, Theorem 6.5]. Roughly speaking, after we regularize the hypergraph under consideration, we need good estimates on the number of hyperedges a polyad can contain. For this we apply some form of a counting lemma proven in [KRS02, Theorem 6.5] ("dense counting lemma") to an equitable family of partitions, in particular the last "layer" of this family forms a very regular partition of the $(k-1)$-subsets with precision δ. Also note, that choosing the function δ smaller does not affect the conclusion of Theorem 2.17.

Now, let m_0 be given by Theorem 2.17 and t_0 by Theorem 2.16. Further we choose n_0 larger than $t_0 \cdot m_0$ and another n_0 given by Theorem 2.16.

Consider a hypergraph H on $n \geq n_0$ vertices. We assume without loss of generality that $t_0!$ divides n, as otherwise, adding less than $t_0!$ isolated vertices, we obtain a super-hypergraph $H' \supset H$ and we prove the asymptotic statement for H', which yields then the claim for H immediately.

So fix any r-hyperedge-coloring φ of H, without a monochromatic subhypergraph F, and denote by H_{col} the hyperedges of H colored by the color $\mathrm{col} \in [r]$. Apply Theorem 2.16 with the parameters k, $c = r$, δ_k, η and the functions f and δ specified above. We obtain from Theorem 2.16 an integer t_0 and a family of partitions $\mathscr{P} = \mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$ such that the properties specified in Theorem 2.16 hold. Roughly speaking, we know that H_{col} is $(\delta_k, *, f)$-regular with respect to the obtained family of partitions for every color col.

We discard from our consideration the following colored hyperedges in H.

- all hyperedges which are not in $Cross_k(\mathscr{P}^{(1)})$, which are at most $\eta\binom{n}{k}$, and
- all hyperedges which are contained in $(\delta_k, *, f(\boldsymbol{a}^{\mathscr{P}}))$-irregular polyads with respect

to one of the colors, hence at most

$$r\delta_k |V|^k = r\delta_k n^k$$

such hyperedges, and

- furthermore, for every color we discard all hyperedges that are contained in $(\delta_k, *, f(\boldsymbol{a}^{\mathscr{P}}))$-regular polyads of density less than ξ, which are at most $r\xi\binom{n}{k}$.

So, in total we discard at most

$$\eta\binom{n}{k} + r\delta_k n^k + r\xi\binom{n}{k} < 2r\xi n^k \tag{5.111}$$

hyperedges, where we used (5.108) and (5.109).

There are

$$N_p := \binom{a_1}{k} \cdot \prod_{i=2}^{k-1} a_i^{\binom{k}{i}} \tag{5.112}$$

$(k, k-1)$ polyads in the partition $\mathscr{P}(k-1, \boldsymbol{a}^{\mathscr{P}})$. Due to the choice of δ, in every polyad $\boldsymbol{\mathcal{P}}(J)$ there are at most

$$E_p^+ := (1 + \varepsilon/4)\left(\frac{n}{a_1}\right)^k \Big/ \prod_{i=2}^{k-1} a_i^{\binom{k}{i}} \tag{5.113}$$

many hyperedges in each of the r colors, due to [KRS02, Theorem 6.5].

We define p_{col} to be the number of $(\delta_k, *, f(\boldsymbol{a}^{\mathscr{P}}))$-regular polyads of density at least ξ in the color col, while e_j for $j \in [r]$ denotes the number of $(\delta_k, *, f(\boldsymbol{a}^{\mathscr{P}}))$-regular polyads of density at least ξ in *exactly* j colors. Furthermore we know that every "monochromatic" slice cannot have more than $\mathrm{ex}(a_1, F)$ such regular polyads, as otherwise, the counting lemma, Theorem 2.17, would imply that the hypergraph H contains a monochromatic copy of F which contradicts our choice of the coloring of the set of hyperedges of H.

Note that there are exactly

$$S := \prod_{i=2}^{k-1} a_i^{\binom{a_1}{i}}$$

different slices (see Section 2.4.5), while every polyad occurs in exactly

$$S \cdot \prod_{i=2}^{k-1} a_i^{-\binom{k}{i}}$$

many slices.

Thus, we infer by averaging for every color $\mathrm{col} \in [r]$ that the number p_{col} of polyads satisfies

$$p_{\text{col}} \leq \text{ex}(a_1, F) \cdot \prod_{i=2}^{k-1} a_i^{\binom{k}{i}}. \tag{5.114}$$

On the other hand, the following simple identity holds

$$\sum_{s=1}^{r} s \cdot e_s = \sum_{\text{col: all } r \text{ colors}} p_{\text{col}} \overset{(5.114)}{\leq} r \cdot \text{ex}(a_1, F) \cdot \prod_{i=2}^{k-1} a_i^{\binom{k}{i}}. \tag{5.115}$$

The number of r-colorings of the hyperedges of H, that yield the given family of partitions $\mathscr{P}$ and the colored polyads, can be bounded from above by

$$\binom{\binom{n}{k}}{2r \cdot \xi \cdot n^k} \cdot r^{2r\xi n^k} \cdot \left(\prod_{s=1}^{r} s^{e_s}\right)^{E_p^+} \leq 2^{h(2k!r\xi)n^k/k!} \cdot r^{2r\xi n^k} \cdot \left(\prod_{s=1}^{r} s^{e_s}\right)^{E_p^+}. \tag{5.116}$$

Since

$$\sum_{s=1}^{r} e_s \leq \binom{a_1}{k} \cdot \prod_{i=2}^{k-1} a_i^{\binom{k}{i}}$$

we may view $\prod_{s=1}^{r} s^{e_s}$ as a product of at most $\binom{a_1}{k} \cdot \prod_{i=2}^{k-1} a_i^{\binom{k}{i}}$ factors. The sum of those factors equals $\sum_{s=1}^{r} s \cdot e_s$, which due to (5.115) is bounded from above by

$$r \cdot e^{\varepsilon/5} \cdot \pi_F \cdot \binom{a_1}{k} \cdot \prod_{i=2}^{k-1} a_i^{\binom{k}{i}}.$$

Since a product of positive reals with bounded sum of the factors is maximized, when all factors are equal, one can show that

$$\prod_{s=1}^{r} s^{e_s} \leq \left(e^{\varepsilon/5} \cdot \pi_F \cdot r\right)^{\binom{a_1}{k} \cdot \prod_{i=2}^{k-1} a_i^{\binom{k}{i}}} \quad \text{if} \quad e^{\varepsilon/5} \cdot \pi_F \cdot r \geq e\,, \tag{5.117}$$

and

$$\prod_{s=1}^{r} s^{e_s} \leq e^{(r/e) \cdot e^{\varepsilon/5} \cdot \pi_F \binom{a_1}{k} \cdot \prod_{i=2}^{k-1} a_i^{\binom{k}{i}}} \quad \text{if} \quad e^{\varepsilon/5} \cdot \pi_F \cdot r < e\,, \tag{5.118}$$

see, e.g., [ABKS04, Lemma 4.3].

We upper bound the number of different $(\eta, \delta(\boldsymbol{a}^{\mathscr{P}}), \boldsymbol{a}^{\mathscr{P}})$-equitable families of partitions which are t_0-bounded together with $(\delta_k, *, f(\boldsymbol{a}^{\mathscr{P}}))$-regular polyads in every color of density at least ξ by

$$\left(\prod_{i=1}^{k-1} t_0^{\binom{n}{i}}\right) \cdot 2^{r\cdot\binom{a_1}{k}\cdot\prod_{i=2}^{k-1} a_i^{\binom{k}{i}}} \leq t_0^{2n^{k-1}}. \tag{5.119}$$

With (5.116) and (5.119) we obtain for $e^{\varepsilon/5} \cdot \pi_F \cdot r \geq e$:

$$\begin{aligned}
c_{r,F}(n) &\leq t_0^{2n^{k-1}} \cdot 2^{h(2k!r\xi)n^k/k!} \cdot r^{2r\xi n^k} \cdot \left(e^{\varepsilon/5} \cdot \pi_F \cdot r\right)^{\binom{a_1}{k}\cdot\prod_{i=2}^{k-1} a_i^{\binom{k}{i}}(E_p^+)} \\
&\overset{(5.113)}{\leq} t_0^{2n^{k-1}} \cdot 2^{h(2k!r\xi)n^k/k!} \cdot r^{2r\xi n^k} \cdot \left(e^{\varepsilon/5} \cdot \pi_F \cdot r\right)^{(1+\varepsilon/4)\binom{n}{k}} \\
&\overset{(5.113)}{\leq} t_0^{2n^{k-1}} \cdot (\pi_F \cdot r)^{\varepsilon n^k/2} \cdot \left(e^{\varepsilon/5} \cdot \pi_F \cdot r\right)^{(1+\varepsilon/4)\binom{n}{k}} \\
&\overset{(5.110)}{\leq} (\pi_F \cdot r)^{\binom{n}{k}+\varepsilon n^k} = (\pi_F \cdot r)^{\binom{n}{k}+o(n^k)},
\end{aligned} \tag{5.120}$$

as $\varepsilon > 0$ can be chosen arbitrary small.

Similarly, with (5.116) and (5.119) we obtain for $\pi_F \cdot r < e$:

$$e^{(r/e)(\pi_F+o(1))\binom{n}{k}}$$

□

5.5 Concluding remarks

In this chapter we studied a problem asking for the maximum possible number of colorings of the hyperedges of a hypergraph with r colors without creating a monochromatic copy of some fixed hypergraph F. For 2 colors this number is easily seen to be bounded by $|\mathrm{Forb}(n, F)|$. We proved a rather general structural result about those hypergraphs H that achieve many (at least $2^{\mathrm{ex}(n,F)-o(n^k)}$) restricted edge colorings without monochromatic copy of F, under the assumption of s-stability for F, see Theorem 1.11. We applied strong hypergraph regularity lemma, Theorem 2.16. This approach has many common features with the study of $\mathrm{Forb}(n, F)$ from Chapter 4. For example, we believe that we can extend the result of Balogh and Mubyai [BMa] and show that almost all 4-uniform hypergraphs without a copy of the 4-uniform generalized triangle T_4 are 4-partite.

Also, equipped with our structural theorem, Theorem 1.11, one might be able to exactly determine the function $c_{r,F}(n)$ for various other hypergraphs F. Such natural candidates are the (1-stable) hypergraphs from [FPS05, KS05a, FPS06, FMP08], where also extremal hypergraphs are known. There it is plausible that $c_{r,F}(n) = r^{\mathrm{ex}(n,F)}$ for $r = 2$ or 3 and n large.

5.5.1 Forbidden 2-colorings of the Fano plane

The following generalization of $c_{2,K_\ell}(n)$ for graphs was studied by Balogh [Bal06]. For a fixed k-uniform hypergraph F, an integer r, and an r-coloring χ of the hyperedges of F, which uses all r colors, we denote for a k-uniform hypergraph H by $c_{r,\chi,F}(H)$ the number of colorings of the set of hyperedges H with r colors which do not contain a copy of F that is identical to χ up to permutation of the color classes. We call such colorings of H (χ, F)-free. Similarly, as before we set $c_{r,\chi,F}(n) = \max c_{r,\chi,F}(H)$, where the maximum runs over all k-uniform hypergraphs on n vertices.

Balogh [Bal06] showed that $c_{2,\chi,K_\ell}(n) = 2^{\mathrm{ex}(n,K_\ell)}$. On the other hand, for three colors ($r = 3$), it is easy to see that $c_{3,\chi,K_3}(n) \geq 2^{\binom{n}{2}} \gg 3^{n^2/4}$, since trivially no 2-coloring of K_n admits a triangle with 3 colors. We can prove a similar result for 2-colorings in the special case, when F is the Fano plane.

Theorem 5.13. *For every 2-coloring χ of the hyperedges of the Fano plane F, which uses both colors, there exists an n_0 such that for all $n \geq n_0$ we have $c_{2,\chi,F}(n) = 2^{\mathrm{ex}(n,F)}$ and the only 3-uniform hypergraph H on n vertices with $c_{2,\chi,F}(H) = 2^{\mathrm{ex}(n,F)}$ is B_n.*

The proof of Theorem 5.13 follows the lines of the proof of Theorem 1.12 and we discuss the required adjustments below.

Proof of Theorem 5.13 (sketch). First an analogous extension of Theorem 5.1 is proved. Again the weak hypergraph regularity lemma yields cluster-hypergraphs H_{red} and H_{blue}. Lemma 2.10 implies that for every 2-coloring, which does not contain a χ-colored copy of F, the number $e(H_2)$ of hyperedges which appear in both cluster-hypergraphs satisfies $e(H_2) = |E(H_{\mathrm{red}}) \cap E(H_{\mathrm{blue}})| \leq e(B_t)$, where t is the number of vertex classes of the regular partition. Now a simple calculation (similar to (5.9)–(5.13) shows that if $e(H_2) < (1 - o(1))e(B_t)$ for every (χ, F)-free coloring of H, then this contradicts the assumption that $c_{2,\chi,F}(H) \geq 2^{e(B_n)}$. Thus there must be a (χ, F)-free coloring of H with $e(H_2) \geq (1 - o(1))e(B_t)$. Now the stability theorem for Fano plane-free hypergraphs yields a partition $A\dot{\cup}B = [t]$ with $|E_{H_2}(A) \cup E_{H_2}(B)| = o(t^3)$, however, we still have to bound the number of hyperedges of $H_1 = ([t], E(H_{\mathrm{red}}) \triangle E(H_{\mathrm{blue}}))$, which are completely contained in A or B. For that we note that $E(H_1) \cup E(H_2)$ cannot contain a copy of F with precisely one hyperedge in $E(H_1)$. Since then again Lemma 2.10 yields a copy of F which has the same coloring as χ. (Here we use the assumption that χ is indeed not a monochromatic coloring of F.) But since $e_{H_2}(A, B) \geq (1 - o(1))e(B_t)$ this implies $e_{H_1}(A) + e_{H_1}(B) \leq o(t^3)$ by a simple counting argument, which gives the appropriate extension of Theorem 5.1.

In the second part, one follows the arguments from Theorem 1.12. Again the proof goes by induction and we show that if $c_{2,\chi,F}(H) \geq 2^{e(B_n)+m}$ and $H \neq B_n$ then there exists a subhypergraph H' on $n' \geq n - 3$ vertices such that $c_{2,\chi,F}(H') \geq 2^{e(B_{n'})+m+1}$. The proof follows the lines of Theorem 1.12 (adjusted for the case $r = 2$). We only have to change the definition of the set $\mathcal{C}_1$ in Case 1. Here we let $\mathcal{C}_1$ be those (χ, F)-free colorings of H such that the link graph L'_Y of v contains many $(\gamma n^2/3)$ blue and L'_X contains many red edges or vice versa. With this adjustment the proof is verbatim. $\square$

Observe that Theorem 1.15 can also be extended to this setting. More precisely, $c_{r,\chi,F} \gg r^{e(B_n)}$ for $r = 4$. In fact, similar to the example of Balogh for K_3 above, we have $c_{r,\chi,F}(n) \geq (r-1)^{\binom{n}{3}} \gg r^{e(B_n)}$ for $r \geq 4$.

This leaves the case $r = 3$ open. However, the similar question is also open for graphs F with more than 3 edges, e.g., to our knowledge it is not known whether $c_{3,\chi,K_4}(n) \gg 3^{2n^3/3}$ or if equality holds.

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Acknowledgment

First and foremost I would like to thank my advisor Mathias Schacht without whom this thesis would not have been possible. I thank him for his guidance, for working together, for fruitful discussions about mathematics and beyond.

I also thank all my coauthors: David Conlon, Hiệp Hàn, Hanno Lefmann, Henry Liu, Lale Özkahya, Vojtěch Rödl and Mathias Schacht. I am especially indebted to Ameera Chowdhury who read and commented on some parts of this thesis.

For work a friendly environment was always important to me, so I would like to heartily thank all my colleagues in the group "Algorithms and Complexity" at the Humboldt-Universität zu Berlin.

I also would like to express my deep gratitude to Anusch Taraz, who introduced me to the exciting field of extremal combinatorics.

I greatly acknowledge the Institute for Pure and Applied Mathematics at UCLA, which I visited several times during my PhD. This would not have been possible without a financial support through NSF.

During my time at the Humboldt-Universität zu Berlin, I was supported by GIF grant no. I-889-182.6/2005.

Finally, I thank my family, my friends and my girlfriend for their support of my mathematical activities.

Erklärung

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Berlin, den 23. Juni 2010,

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